Single-Crossing Differences in Convex Environments*

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Abstract

An agent's preferences depend on an ordered parameter or type. We characterize the set of utility functions with single crossing differences (SCD) in *convex environments*. These include preferences over lotteries, both in expected utility and rank-dependent utility frameworks, and preferences over bundles of goods and over consumption streams. Our notion of SCD does not presume an order on the choice space. This unordered SCD is necessary and sufficient for "interval choice" comparative statics. We present applications to cheap talk, observational learning, and collective choice, showing how convex environments arise in these problems and how SCD/interval choice are useful. Methodologically, our main characterization stems from a result on linear aggregations of single-crossing functions.

Keywords: monotone comparative statics, choice among lotteries, interval equilibria, aggregating single crossing

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1. Introduction

1.1. Overview

Single-crossing properties and their implications for choices are at the heart of many economic models, as highlighted by Milgrom and Shannon (1994). Consider a utility function $u(a, \theta)$, where a is the choice object and θ a preference parameter. In this paper, we completely characterize the structure of $u(\cdot)$ when it has *single-crossing differences (SCD) in a convex environment*. Before defining this property more precisely and explaining our results, we begin with some background and motivation for our work in the context of a leading example of a convex environment: choice among lotteries.

Motivation. In various applications of single crossing, choices have been restricted to deterministic outcomes when it would be desirable to accommodate lotteries. For example:

1. In the canonical cheap-talk model (Crawford and Sobel, 1982), a sender with private type $\theta \in \Theta \subseteq \mathbb{R}$ chooses a costless message to send to a receiver, who then takes a decision a. A single-crossing property on the sender's utility $u(a, \theta)$ ensures that any equilibrium is an "interval equilibrium", i.e., Θ is partitioned into intervals that induce the same decision. This result is predicated on assumptions ensuring that in equilibrium, the sender can anticipate exactly what decision is induced by any message. However, for many applications one would like to permit the sender to have uncertainty about the receiver's preferences; but then, each message would in fact induce some lottery over decisions.

2. In voting models, a voter indexed by θ has preferences over outcomes a given by the utility function $u(a, \theta)$. A single-crossing property of $u(a, \theta)$ guarantees a well-behaved majority preference and the existence of a Condorcet winner (Gans and Smart, 1996). This result is central to various political competition models à la Downs (1957). But the presumption here is that voters or political candidates choose directly among final outcomes. More realistically, the relevant choice is only among some set of policies whose outcomes are uncertain at the time of voting.

A challenge with extending single-crossing properties to choice among lotteries is that there is no natural order on the set of all lotteries. Moreover, it is not always apparent a priori what restrictions are reasonable on the set of lotteries facing an agent, in particular whether some form of stochastic dominance can order every choice set they may face. For example, in the cheap-talk problem, the nature of the receiver's preferences may well imply that a higher message (in equilibrium) induces lotteries that have both higher mean and higher variance. Or, in the voting context, the economic and political outcomes are multidimensional and so the set of lotteries is unlikely to comply with standard orders. In other applications, the lotteries may be the result of still further interactions—e.g., they may represent continuations in dynamic strategic problems—which are frequently intractable to structure ex ante.

An alternative approach is to require that an agent's utility difference between *any* pair of lotteries—or, more abstractly, any pair of choice objects—is single crossing in the agent's type or preference parameter. (For any ordered set Θ, a function $f : \Theta \to \mathbb{R}$ is single crossing if its sign is monotonic.) It is this single-crossing property that we study, which we refer to as *single-crossing differences* (SCD); our notion is closely related to that of Milgrom and Shannon (1994). As we will explain, SCD characterizes "interval choice", a fundamental choice property for applications, including the aforementioned cheap-talk and voting problems. But a key question is: for an expected-utility agent, which (Benoulli) utility functions assure SCD over lotteries? It is not enough that SCD holds over pure outcomes:

Example. Let $\Theta = [-1, 1]$, $A = \{0, 1, 2\}$, and $u(a, \theta) = a$ for $a \neq 1$ while $u(1, \theta) = \theta^2 + 1/2$. For any $a, a' \in A$, $u(a, \theta) - u(a', \theta)$ is single-crossing in θ as its sign does not depend on θ. But for G the uniform lottery over actions 0 and 2, the expected utility difference $u(1, \theta) - \mathbb{E}_{a\sim G}[u(a, \theta)] = \theta^2 + 1/2 - (1/2)2 = \theta^2 - 1/2$ is not single crossing in θ. Hence, u has SCD over pure outcomes but does not have SCD over lotteries.

SCD in Convex Environments. Rather than focusing only on choice among lotteries, we take a more general and, we believe, insightful viewpoint. Consider a utility function $u(a, \theta)$, where $a \in A$ is an action (i.e., the choice object) and $\theta \in \Theta$ the ordered type (preference parameter). Say that the choice environment is *convex* if the set of functions $\{u(a, \cdot) : \Theta \to \mathbb{R}\}_{a\in A}$ is convex.[1] Note that this convexity is a property of the utility space, not per se about the structure of the action space.

Convex choice environments abound. The function u can be expected utility and A the set of lotteries on any outcome space. But, as explained by Example 2 in Section 3, u can also be rank-dependent utility (Quiggin, 1982). Or, as detailed in Example 3, A can be a product set representing different dimensions of the choice object, and u can be a multidimensional utility function with convex range. This class captures examples of deterministic settings in mechanism design, dynamic consumption streams, and choices over bundles of goods or products with multiple characteristics.

[1] That is, for all $a, a' \in A$ and $\lambda \in (0, 1)$, there is $a'' \in A$ such that for all $\theta \in \Theta$, $u(a'', \theta) = \lambda u(a, \theta) + (1 - \lambda)u(a', \theta)$.

Our paper's main result, Theorem 2, is a characterization of SCD in any convex environment. Theorem 2 establishes that in a convex environment, u has SCD if and only if

$$u(a, \theta) = g_1(a) f_1(\theta) + g_2(a) f_2(\theta) + h(\theta), \tag{1}$$

where f_1 and f_2 are single-crossing functions that satisfy a *ratio-ordering* property we define in Section 3. Roughly speaking, ratio ordering requires that the relative importance placed on $g_1(\cdot)$ versus $g_2(\cdot)$ changes monotonically with type.[2] The idea is transparent when Θ with a minimum $\underline{\theta}$ and a maximum $\overline{\theta}$. Then, u having SCD is equivalent to the existence of a (type-dependent) representation $\tilde{u}(a, \theta)$ that satisfies

$$\tilde{u}(a, \theta) = \lambda(\theta)\tilde{u}(a, \overline{\theta}) + (1 - \lambda(\theta))\tilde{u}(a, \underline{\theta}),$$

where $\lambda : \Theta \to [0, 1]$ is increasing (Proposition 1). In other words, in such a convex environment, SCD is equivalent to being able to represent each type's preferences by a utility function that is a convex combination of those of the extreme types, with higher types putting more weight on the highest type's utility.

In the context of expected utility, there are canonical (Benoulli) functional forms that induce SCD over lotteries: in mechanism design and screening, $u((q, t), \theta) = \theta q - t$ (where $q \in \mathbb{R}$ is quantity, $t \in \mathbb{R}$ is a transfer, and $\theta \in \mathbb{R}$ is the agent's marginal rate of substitution); in optimal delegation without transfers, $u((q, t), \theta) = \theta q + g(q) - t$ (where $q \in \mathbb{R}$ is the allocation, $t \in \mathbb{R}_+$ is money burning, and $\theta \in \mathbb{R}$ is the agent's type; cf. Amador and Bagwell (2013)); in communication/delegation and voting, $u(a, \theta) = -(a - \theta)^2 = 2\theta a - a^2 - \theta^2$ (where $a \in \mathbb{R}$ is an outcome and $\theta \in \mathbb{R}$ is the agent's bliss point). On the other hand, our characterization also makes clear that SCD is quite stringent. For example, within the class of power loss functions, only the quadratic loss function generates SCD over lotteries (Corollary 2). Outside of expected utility, we explain in Subsection 3.2 when discounted utility and Cobb-Douglas utility satisfy SCD in a convex environment; in particular, it holds for the simple two-good case of $u((x, y), \theta) = \theta \log x + (1 - \theta) \log y$, where $x, y > 0$ are the quantities and θ parameterizes the marginal rate of substitution.

Interval Choice and Comparative Statics. As mentioned earlier, SCD characterizes interval choice, which we view as a fundamental choice property. More precisely, say that there is *interval choice* if, no matter the choice set $S \subseteq A$, the set of types choosing any $a \in S$ is an

[2] In particular, if either $f_1(\cdot)$ or $f_2(\cdot)$ is strictly positive, then ratio ordering reduces to saying that the ratio of the two functions is monotonic. More generally, Lemma 1 establishes that ratio ordering is necessary and sufficient for all linear combinations of two single-crossing functions to be single crossing.

interval:

$$(\forall S \subseteq A)\ (\forall \theta_l < \theta_m < \theta_h) \quad a^* \in \bigcap_{\theta \in \{\theta_l, \theta_h\}} \arg\max_{a \in S} u(a, \theta) \implies a^* \in \arg\max_{a \in S} u(a, \theta_m).$$

Theorem 1 in Section 2 shows that, modulo some details, this property is equivalent to u having SCD.

It is intuitive that interval choice and SCD are related to monotone comparative statics. We establish a precise connection in Theorem 4: SCD is equivalent to the existence of an order on A such that

$$(\forall S \subseteq X)\ (\forall \theta_l < \theta_h) \quad \arg\max_{a \in S} u(a, \theta_h) \succeq_{SSO} \arg\max_{a \in S} u(a, \theta_l). \tag{2}$$

Here, $\succeq_{SSO}$ is the strong set order generated by the order on A.

SCD differs from Milgrom and Shannon's (1994) single-crossing property because we do not take as given an exogenous order on the choice space A. Rather, SCD is necessary and sufficient for the existence of *some* order that generates the choice monotonicity in (2). Beyond interval choice, there are other familiar implications of single-crossing properties that in fact rely only our SCD's order-independent notion; for example, SCD is the key to guaranteeing that local incentive compatibility implies global incentive compatibility.[3]

Applications. Section 4 applies SCD in convex environments to three economic problems, highlighting the implications of interval choice. Among other things, the cheap-talk application with uncertain receiver preferences in Subsection 4.1 demonstrates concretely how choices from *all* lotteries emerge naturally. Subsection 4.2 provides an application to observational learning in which a convex environment stems from multidimensional utility. Subsection 4.3 considers collective choice over lotteries, showing how our results contribute to a long-standing question of whether equilibrium exists when political candidates can offer lottery platforms (e.g., Zeckhauser, 1969; Shepsle, 1972).

1.2. An Intuition

A key step towards our characterization in Theorem 2 of SCD in convex environments is establishing that every type's utility over actions is an affine combination of two (type-independent) functions: Equation 1. We can provide a succinct intuition here. Suppose Θ

[3] We mean in a sense analogous to Carroll (2012, Proposition 4). While Carroll establishes his result by defining a single-crossing property with respect to some given order over alternatives, essentially the same logic applies with our order-independent notion of strict SCD (Definition 2).

has a minimum $\underline{\theta}$ and a maximum $\overline{\theta}$. It suffices to show that there are three actions, a_1, a_2, and a_3, such that any type θ's utility from any action a satisfies

$$u(a,\theta) = \lambda_1(a)u(a_1,\theta) + \lambda_2(a)u(a_2,\theta) + \lambda_3(a)u(a_3,\theta), \tag{3}$$

for some $\lambda(a) \equiv (\lambda_1(a), \lambda_2(a), \lambda_3(a))$ with $\sum_{i=1}^{3} \lambda_i(a) = 1$. (Equation 1 follows by setting, for $i = 1, 2$, $f_i(\theta) = u(a_i,\theta) - u(a_3,\theta)$, $g_i(a) = \lambda_i(a)$, and $h(\theta) = u(a_3,\theta)$.) The desired $\lambda(a)$ is the solution to

$$\begin{bmatrix} u(a,\underline{\theta}) \\ u(a,\overline{\theta}) \\ 1 \end{bmatrix} = \begin{bmatrix} u(a_1,\underline{\theta}) & u(a_2,\underline{\theta}) & u(a_3,\underline{\theta}) \\ u(a_1,\overline{\theta}) & u(a_2,\overline{\theta}) & u(a_3,\overline{\theta}) \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1(a) \\ \lambda_2(a) \\ \lambda_3(a) \end{bmatrix},$$

which exists at least when there are three actions for which the 3×3 matrix on the right-hand side is invertible. To interpret this matrix equation, note that in a convex environment, any convex combination of utilities from $\{a_1, a_2, a_3, a\}$ is the utility from some action. Hence, the equation says that one can find two distinct actions with utilities equal to convex utility combinations of $\{a_1, a_2, a_3, a\}$ such that the lowest and highest types are both indifferent between those two actions.[4] By SCD, *all* types must be indifferent between these two actions, which amounts to Equation 3.

A second key step towards Theorem 2 is showing that the two type-independent utilities in Equation 1 must be ratio ordered. The intuition for this step is deferred to Subsection 3.3.

1.3. Related Literature

We now offer a summary of related work, supplying additional details later.

Quah and Strulovici (2012) consider an expected-utility agent choosing under uncertainty about her preferences, which depend on some unknown "state". They ask when single-crossing differences in the Milgrom and Shannon (1994) sense is preserved regardless of the distribution of the state. In our expected-utility application, we consider an agent who has no uncertainty about her preferences but chooses among lotteries. Although these are conceptually different questions, there is a mathematical connection in portions of our analysis. Quah and Strulovici's (2012) question concerns when single crossing is preserved by positive linear combinations of single-crossing functions. On the other hand, our problem turns on *arbitrary* linear combinations of single-crossing functions preserving single crossing. As elaborated after Lemma 1, this explains the difference between Quah and

[4] Take one utility combination to be that corresponding to the uniform distribution $P = (1/4, 1/4, 1/4, 1/4)$ on $\{a_1, a_2, a_3, a\}$ and the other corresponding to $P + \varepsilon(\lambda_1(a), \lambda_2(a), \lambda_3(a), -1)$ for any sufficiently small $\varepsilon > 0$.

Strulovici (2012)'s key condition, signed-ratio-monotonicity, and our condition of ratio ordering. Moreover, there is no analog in their analysis to the linear dependence we deduce in Proposition 2, which is crucial to the functional form (6) in our main characterization.

When $\Theta \subseteq \mathbb{R}$, the utility specification $u(a, \theta) = \theta g_1(a) + g_2(a)$ induces expected utility with SCD over lotteries; indeed, the expected utility difference between any two lotteries is monotonic in θ. The usefulness of this utility specification (or slight variants) to structure choices among arbitrary lotteries has been highlighted by Duggan (2014), Celik (2015), and Kushnir and Liu (2018). In Subsection 5.1, we show how, in convex environments that satisfy a reasonable additional condition, SCD preferences always have such a "monotonic differences" representation. This is a striking consequence of convex environments, as we are not aware of any such result more generally.

In the operations research literature, there has been interest in functional forms for "multi-attribute utility functions" (e.g., Fishburn, 1974). When there are two attributes and the agent has expected-utility preferences, Abbas and Bell (2011) study a "one-switch condition" that is akin to SCD over lotteries on one attribute. In that setting, they offer a result related to our Proposition 1. However, they do not identify ratio ordering as the property that characterizes when single crossing is preserved under aggregation, which is a key contribution of our analysis (Lemma 1 and Proposition 2). Also novel to our paper are the comparative statics characterizations of SCD (Theorem 1 and Theorem 4) and our economic applications.

For choice among restricted sets of lotteries (which we wish to largely avoid, for reasons mentioned earlier), there are various prior results on the conditions for expected-utility preferences to deliver comparative statics. Standard restricted classes of lotteries include those ordered by first-order stochastic dominance (Topkis, 1978) and likelihood-ratio dominance (Karlin, 1968; Athey, 2002).[5]

Outside of expected utility on lottery spaces, we are not aware of any work highlighting SCD in convex environments. In our view, the lens of convex environments is a significant contribution of our paper.

2. Single-Crossing Differences and Interval Choice

Our analysis begins by formalizing comparative statics results that justify a notion of single-crossing differences without reference to an order over the choice space.

[5] For first-order stochastic dominance, the requirement is that the utility function is supermodular; for likelihood-ratio dominance, it is log-supermodularity. Smith (2011) considers choice among arbitrary lotteries and their certainty equivalents.

Let $(\Theta, \leq)$ be a (partially) ordered set containing upper and lower bounds for all pairs.[6] We often refer to elements of Θ as *types*.

Definition 1. A function $f : \Theta \to \mathbb{R}$ is:

1. *single crossing* (resp., from below or from above) if $\text{sign}[f]$ is monotonic (resp., increasing or decreasing);[7]
2. *strictly single crossing* if it is single crossing and there are no $\theta' < \theta''$ such that $f(\theta') = f(\theta'') = 0$.

Definition 2. Given any set A, a function $u : A \times \Theta \to \mathbb{R}$ has:

1. *single-crossing differences (SCD)* if $\forall a, a' \in A$, the difference $D_{a,a'}(\theta) \equiv u(a, \theta) - u(a', \theta)$ is single crossing in θ;
2. *strict single-crossing differences (SSCD)* if $\forall a, a' \in A$ such that $a \neq a'$, $D_{a,a'}(\theta)$ is strictly single crossing in θ.

Our definition of (S)SCD is related to but different from Milgrom and Shannon (1994), who stipulate that $u : A \times \Theta \to \mathbb{R}$ has (strict) single-crossing differences given an order $\succeq$ on A if for all $a' \succ a''$, $D_{a',a''}(\theta)$ is (strictly) single crossing from below. (Here, $\succ$ is the strict component of $\succeq$. Note that the (S)SCD terminology is due to Milgrom (2004).) We do not presume that A is ordered, but we consider differences for all pairs of elements of A. If A is completely ordered, then our definition is weaker than that of Milgrom and Shannon (1994) because ours does not constrain the direction of single crossing. As established below, our notion characterizes related but distinct comparative statics from theirs.

Interval Choice. We say that $\Theta_0 \subseteq \Theta$ is an *interval* if $\theta_l, \theta_h \in \Theta_0$ and $\theta_l < \theta_m < \theta_h$ imply $\theta_m \in \Theta_0$. Let $C : 2^A \times \Theta \rightrightarrows A$ with $C(S, \theta) \subseteq S$ for each $S \subseteq A$ and $\theta \in \Theta$. We say that C has *interval choice* if $\{\theta : a \in C(S, \theta)\}$ is an interval for each $S \subseteq A$ and $a \in S$. That is, interpreting C as a choice correspondence, the set of types choosing any option given any choice set is an interval. We say that $u : A \times \Theta \to \mathbb{R}$ *strictly violates SCD* if there are $a, a' \in A$ and $\theta_l < \theta_m < \theta_h$ such that $\min\{D_{a,a'}(\theta_l), D_{a,a'}(\theta_h)\} > 0 > D_{a,a'}(\theta_m)$.

[6] A partial order—hereafter, also referred to as just an order—is a binary relation that is reflexive, anti-symmetric, and transitive (but not necessarily complete). An upper (resp., lower) bound of $\Theta_0 \subseteq \Theta$ is $\overline{\theta} \in \Theta$ (resp., $\underline{\theta} \in \Theta$) such that $\theta \leq \overline{\theta}$ (resp., $\underline{\theta} \leq \theta$) for all $\theta \in \Theta_0$. While none of our results require any assumptions on the cardinality of Θ, the results in Section 3 are trivial when $|\Theta| < 3$. See an earlier version of this paper, Kartik, Lee, and Rappoport (2019, Appendix I), for how our results extend when $(\Theta, \leq)$ is only a pre-ordered set, i.e., when $\leq$ does not satisfy anti-symmetry.

[7] For $x \in \mathbb{R}$, $\text{sign}[x] = 1$ if $x > 0$, $\text{sign}[x] = 0$ if $x = 0$, and $\text{sign}[x] = -1$ if $x < 0$. A function $h : \Theta \to \mathbb{R}$ is increasing (resp., decreasing) if $\theta_h > \theta_l \implies h(\theta_h) \geq h(\theta_l)$ (resp., $h(\theta_h) \leq h(\theta_l)$); it is monotonic if it is either increasing or decreasing. An equivalent, and perhaps more familiar, definition of f being single crossing from below is $(\forall \theta < \theta')\ f(\theta) \geq (>) 0 \implies f(\theta') \geq (>) 0$.

Theorem 1. *Let $u : A \times \Theta \to \mathbb{R}$ and $C_u(S,\theta) \equiv \arg\max_{a\in S} u(a,\theta)$ for any $S \subseteq A$ and θ.*

1. *If u has SCD, then the choice correspondence C_u has interval choice. If u strictly violates SCD, then C_u does not have interval choice.*
2. *If $|\Theta| \geq 3$, then u has SSCD if and only if every selection from C_u has interval choice.*

The intuition for the sufficiency of (S)SCD in Theorem 1 is straightforward. Regarding necessity, we note that a violation of SCD—as opposed to a strict violation—is compatible with the choice correspondence having interval choice: e.g., $A = \{a', a''\}$, $\Theta = \{\theta_l, \theta_m, \theta_h\}$ with $\theta_l < \theta_m < \theta_h$, and $\min\{D_{a',a''}(\theta_l), D_{a',a''}(\theta_h)\} > 0 = D_{a',a''}(\theta_m)$.[8] In Part 2 of the Theorem, if $|\Theta| = 2$ then any selection from any choice correspondence trivially has interval choice, yet u does not have SSCD when $D_{a,a'}(\theta) = 0$ for some a, a' and all θ.

Monotone Comparative Statics. Our choice space A is unordered. Intuitively, interval choice is intimately related to there being monotone comparative statics (MCS)—i.e., in some sense, higher types make higher choices—with respect to *some* complete order on the choice space. We formalize this connection in Subsection 5.2 by tying (S)SCD to such MCS. In brief, given any order on A, we order subsets of A by the corresponding strong set order, and say that the function u has MCS if, no matter the choice set $S \subseteq A$ and types $\theta_h > \theta_l$, the higher type chooses a higher set: $C_u(S,\theta_h) \geq C_u(S,\theta_l)$. Roughly speaking, Theorem 4 in Subsection 5.2 shows that an order on A induces MCS if and only if u has SCD and the order is a refinement of a natural "SCD-order" generated by u.

3. Single-Crossing Differences in Convex Environments

This section characterizes single-crossing differences in "rich" environments. We now assume the existence of a strictly increasing real-valued function on $(\Theta, \leq)$.[9] This requirement is satisfied, for example, when Θ is finite, or $\Theta \subseteq \mathbb{R}^n$ is endowed with the usual order. We assume the environment (A, Θ, u) is *convex* in the following sense:

$$\text{the set of functions } \{u(a,\cdot) : \Theta \to \mathbb{R}\}_{a\in A} \text{ is convex.} \tag{$\star$}$$

[8] On the other hand, a strict violation of SCD is slightly stronger than needed: one could weaken its requirement to $\min\{D_{a',a''}(\theta_l), D_{a',a''}(\theta_h)\} \geq 0 > D_{a',a''}(\theta_m)$. Our formulation with both inequalities being strict amounts to putting aside indifferences, which proves convenient for the applications in Section 4.

[9] That is, we assume $\exists h : \Theta \to \mathbb{R}$ such that $\underline{\theta} < \overline{\theta} \implies h(\underline{\theta}) < h(\overline{\theta})$. This requirement is related to utility representations for possibly incomplete preferences (Ok, 2007, Chapter B.4.3). A sufficient condition is that Θ has a countable order dense subset, i.e., there is a countable set $\Theta_0 \subseteq \Theta$ such that $(\forall \underline{\theta}, \overline{\theta} \in \Theta \setminus \Theta_0) \quad \underline{\theta} < \overline{\theta} \implies \exists \theta_0 \in \Theta_0$ s.t. $\underline{\theta} < \theta_0 < \overline{\theta}$ (Jaffray, 1975, Corollary 1). The assumption only plays a technical role in establishing our characterization of strict SCD, i.e., in the second statement of Theorem 2.

That is, a convex environment is rich enough insofar as, in utility terms, for any weighted combination of any two actions, there is a third action that replicates the weighted combination. We stress that convexity is in terms of utilities: it is neither necessary nor sufficient that the action space A be convex; indeed, we have not assumed any structure on A.

Example 1 (Expected Utility)**.** Consider an expected-utility agent who chooses among lotteries. There is a set of consequences X and the agent has utility $v(x,\theta)$. Letting $A \equiv \Delta X$ be the set of all finite-support lotteries over X, the agent's utility from lottery $P \in A$ is given by $u(P,\theta) \equiv \int_x v(x,\theta)\mathrm{d}P$.[10] The linearity of u in its first argument and the fact that A is convex readily imply ($\star$). $\square$

Example 2 (Rank-Dependent Expected Utility)**.** Continuing with choice among lotteries, a convex environment can also accommodate non-expected utility frameworks. By virtually the same logic as above for expected utility, it is sufficient for ($\star$) that the utility from lottery P be given by $u(P,\theta) \equiv \int_x v(x,\theta)\mathrm{d}(w \circ P)$, with $w : \Delta X \to \Delta X$ an arbitrary reweighting function whose image is convex. For example, a standard formulation of rank-dependent utility (Quiggin, 1982; Diecidue and Wakker, 2001) corresponds to $X \equiv \{x_1, \ldots, x_n\} \subset \mathbb{R}$ with $x_1 < \cdots < x_n$, and a strictly increasing function $\hat{w} : [0,1] \to [0,1]$ satisfying $\hat{w}(0) = 0$ and $\hat{w}(1) = 1$ such that for any lottery P and consequence x_i, the reweighting is given by $(w \circ P)(x_i) \equiv \hat{w}\left(\sum_{j=1}^{i} p(x_j)\right) - \hat{w}\left(\sum_{j=1}^{i-1} p(x_j)\right)$, where p is the probability mass function of the lottery P. So long as $\hat{w}$ is continuous, w has a convex image as required; the image is simply ΔX. $\square$

Example 3 (Multidimensional Utility)**.** While convexity is naturally induced by lotteries, it is also satisfied by applications in which A contains only deterministic actions. One setting is that of choice among multidimensional actions. Specifically, $A \equiv A_1 \times \ldots \times A_n \subseteq \mathbb{R}^n$ and for any $a \equiv (a_1, \ldots, a_n)$, utility is given by $u(a,\theta) \equiv \sum_{i=1}^n g_i(a_i) f_i(\theta)$ for some pairs of functions $(g_i, f_i)_{i=1}^n$, with each g_i having a convex image. The fact that A is a product set and each g_i has a convex image ensures ($\star$). We refer to this specification as *multidimensional utility*.

Here are some economic contexts in which there is multidimensional utility. First, a consumer chooses among products with multiple characteristics or a bundle of multiple goods, denoted $(a_1, \ldots, a_n)$. Each characteristic or good i with quality or quantity a_i has a common value $g_i(a_i)$, but the tradeoff across characteristics/goods varies with the consumer's preference parameter θ, as given by $f_i(\theta)$. Second, a designer uses an incentive-compatible

[10] We restrict attention to finite-support distributions throughout the paper for ease of exposition, as it guarantees that expected utility is well defined no matter the distribution and utility function. Nevertheless, we write integrals rather than summations when it simplifies notation regarding the domain of integration/summation.

direct mechanism $\varphi : T \to A$ that maps an agent's private type $t \in T \equiv \{1, \ldots, n\}$ to an action $\varphi(t) \in A$. The designer's payoff is $\sum_t v(\varphi(t), t) f(t; \theta)$, where $v(a, t)$ is the designer's utility from allocating a to t, and $f(\cdot; \theta) \in \Delta T$ is a type distribution that depends on some parameter θ. Third, an agent chooses consumption c_t in each period $t \in T \equiv \{1, \ldots, n\}$. The agent's present discounted value is $\sum_t v(c_t) \rho(t; \theta)$, where $\rho(t; \theta)$ is discount function parameterized by θ. $\square$

Example 4 (Experiments)**.** Our final example is one in which an expected-utility agent can only choose from a proper, but convex, subset of lotteries. Let Ω be a finite set of "states". We refer to $\Delta\Omega$ as the set of beliefs or posteriors and $\Delta\Delta\Omega$ as the set of experiments (with finite support). In Bayesian persuasion (Kamenica and Gentzkow, 2011) or, more broadly, information design (Bergemann and Morris, 2019), an expected-utility agent has preferences represented by $v(p, \theta)$, where $p \in \Delta\Omega$ and θ is a preference parameter. The expected utility from experiment $Q \in \Delta\Delta\Omega$ is given by $u(Q, \theta) \equiv \int_p v(p, \theta) \mathrm{d}Q$. If we consider all experiments, then this setting is a special case of Example 1. But given a prior $p^* \in \Delta\Omega$, any experiment must in fact be Bayes-plausible, i.e., its distribution of posteriors must average to the prior p^*. So, given p^*, the agent can only choose an experiment in $A \equiv \{Q \in \Delta\Delta\Omega : \int_{p \in \Delta\Omega} p \mathrm{d}Q = p^*\}$. Nevertheless, the linearity of u in its first argument and the convexity of A still imply ($\star$). Indeed, based on Example 2, Condition ($\star$) will hold even for rank-dependent expected-utility agents choosing among Bayes-plausible experiments. $\square$

3.1. The Characterization

Our characterization of SCD in convex environments (Theorem 2 below) requires the following definition.

Definition 3. Let $f_1, f_2 : \Theta \to \mathbb{R}$ each be single crossing.

1. f_1 *ratio dominates* f_2 if

$$(\forall \theta_l < \theta_h) \quad f_1(\theta_l) f_2(\theta_h) \leq f_1(\theta_h) f_2(\theta_l), \quad \text{and} \tag{4}$$

$$(\forall \theta_l < \theta_m < \theta_h) \quad f_1(\theta_l) f_2(\theta_h) = f_1(\theta_h) f_2(\theta_l) \iff \begin{cases} f_1(\theta_l) f_2(\theta_m) = f_1(\theta_m) f_2(\theta_l), \\ f_1(\theta_m) f_2(\theta_h) = f_1(\theta_h) f_2(\theta_m). \end{cases} \tag{5}$$

2. f_1 *strictly ratio dominates* f_2 if Condition (4) holds with strict inequality.

3. f_1 and f_2 are *(strictly) ratio ordered* if either f_1 (strictly) ratio dominates f_2 or vice-versa.

Condition (4) contains the essential idea of ratio dominance and is what we focus on in the main text; Condition (5) only deals with some knife-edged cases that are discussed in

Appendix B.1. The definition of strict ratio dominance does not make reference to Condition (5) because that condition is vacuous when Condition (4) holds with strict inequality.

Since ratio dominance involves weak inequalities, f_1 can ratio dominate f_2 and vice-versa even when $f_1 \neq f_2$: consider $f_1 = -f_2$. We use the terminology "ratio dominance" because when f_2 is a strictly positive function, Condition (4) is the requirement that the ratio $f_1(\theta)/f_2(\theta)$ must be increasing in θ. Indeed, if both f_1 and f_2 are probability densities of random variables Y_1 and Y_2, then (4) says that Y_1 stochastically dominates Y_2 in the sense of likelihood ratios.[11]

Condition (4) is a natural generalization of the increasing ratio property to functions that may change sign. To get a geometric intuition, suppose f_1 strictly ratio dominates f_2. Let $f(\theta) \equiv (f_1(\theta), f_2(\theta))$. For every $\theta_l < \theta_h$, $f_1(\theta_l)f_2(\theta_h) - f_1(\theta_h)f_2(\theta_l) < 0$ implies that the vector $f(\theta_l)$ moves to $f(\theta_h)$ through a rescaling of magnitude and a clockwise—rather than counterclockwise—rotation (throughout our paper, a "rotation" must be no more than 180 degrees).[12]

Hence, f_1 and f_2 are ratio ordered only if $f(\theta)$ *rotates monotonically* as θ increases, either always clockwise or always counterclockwise.[13] It follows that the set $\{f(\theta) : \theta \in \Theta\}$ must be contained in a closed half-space of $\mathbb{R}^2$ defined by a hyperplane that passes through the origin: otherwise, there will be two pairs of vectors such that an increase in θ corresponds to a clockwise rotation in one pair and a counterclockwise rotation in the other. See Figure 1. In its panel (b), the non-monotonic rotation of $f(\theta)$ is clear when $\Theta = \{\theta_l, \theta_m, \theta_h\}$ with $\theta_l < \theta_m < \theta_h$, but it is also present when $\Theta = \{\theta_l, \theta_m, \hat{\theta}_h\}$ with $\theta_l < \theta_m < \hat{\theta}_h$; in the latter case, the aforementioned half-space requirement is violated.

We can now state our main characterization. For brevity, and to emphasize the convex environment assumption, we write (S)SCD* as shorthand for "In a convex environment, u has (S)SCD…". Furthermore, we say that A is *minimal* if there is no pair of utility-

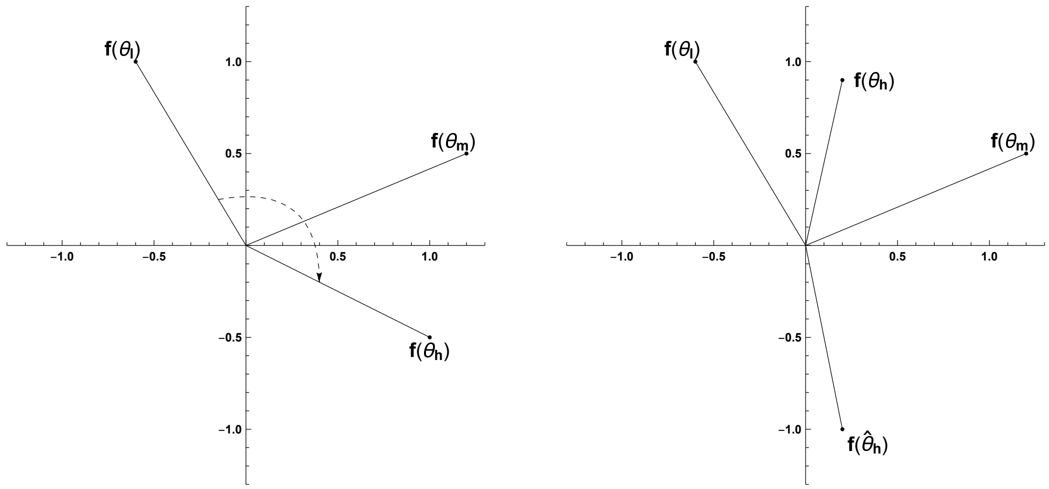
[11] From the viewpoint of information economics, think of θ as a signal of a state $s \in \{1, 2\}$, drawn from the density $f(\theta|s) \equiv f_s(\theta)$. Condition (4) is Milgrom's (1981) monotone likelihood-ratio property for $f(\theta|s)$.

[12] To confirm this point, recall that from the definition of cross product,

$$\begin{aligned}(f_1(\theta_l), f_2(\theta_l), 0) \times (f_1(\theta_h), f_2(\theta_h), 0) &= \|f(\theta_l)\|\|f(\theta_h)\| \sin(r) e_3 \\ &= (f_1(\theta_l)f_2(\theta_h) - f_1(\theta_h)f_2(\theta_l))\, e_3,\end{aligned}$$

where r is the counterclockwise angle from $f(\theta_l)$ to $f(\theta_h)$, $e_3 \equiv (0, 0, 1)$, $\times$ is the cross product, and $\|\cdot\|$ is the Euclidean norm. If $\sin(r) < 0$ (resp., $\sin(r) > 0$), then $f(\theta_l)$ moves to $f(\theta_h)$ through a clockwise (resp., counterclockwise) rotation.

[13] The preceding discussion establishes this point under the presumption of strict ratio ordering; however, because of the hypothesis in Definition 3 that f_1 and f_2 are single crossing and because of Condition (5), the conclusion holds for ratio ordering too. Furthermore, it can be confirmed that a monotonic rotation of $f(\cdot)$ implies ratio ordering if there are no θ' and θ'' such that $f(\theta')$ and $f(\theta'')$ are collinear.



(a) Condition (4) holds for $\theta_l < \theta_m < \theta_h$.

(b) Condition (4) fails both when $\Theta = \{\theta_l, \theta_m, \theta_h\}$ with $\theta_l < \theta_m < \theta_h$, and when $\Theta = \{\theta_l, \theta_m, \hat{\theta}_h\}$ with $\theta_l < \theta_m < \hat{\theta}_h$.

Figure 1: Geometric representation of Condition (4).

indistinguishable actions: $\forall a, a' \in A$, $\exists \theta$ such that $D_{a,a'}(\theta) \neq 0$.

Theorem 2. *The function $u : A \times \Theta \to \mathbb{R}$ has SCD* if and only if it takes the form*

$$u(a, \theta) = g_1(a) f_1(\theta) + g_2(a) f_2(\theta) + h(\theta), \tag{6}$$

with f_1 and f_2 each single crossing and ratio ordered. In addition, if A is minimal, then u has SSCD* if and only if f_1 and f_2 are strictly ratio ordered.[14]

We make a number of observations to help interpret Theorem 2.

The theorem says that for u to have SCD*, it must be possible to write it in the form (6). Notice that given (6), for any $a_0, a_1 \in A$, the function $u(a_1, \cdot) - u(a_0, \cdot)$ is a linear combination of $f_1(\cdot)$ and $f_2(\cdot)$. Therefore, to rule out the possibility of the form (6), it is sufficient to find $a_0, a_1, a_2, a_3 \in A$ and $\theta_l < \theta_m < \theta_h$ such that the 3×3 matrix $M \equiv [u(a_i, \theta_j) - u(a_0, \theta_j)]_{i \in \{1,2,3\}, j \in \{l,m,h\}}$ is invertible. This procedure is often useful to reject SCD*, as we illustrate subsequently in Corollary 2.

Given the functional form (6), not only is SCD* assured by f_1 and f_2 each being single

[14] If A is not minimal, then so long as $|\Theta| > 1$, u violates SSCD* because for some a and a', $D_{a,a'}$ is a zero function and hence not strictly single crossing. Nonetheless, the characterization applies when we consider utility-indistinguishable actions as equivalence classes and the corresponding utility function on those equivalence classes.

crossing and ratio ordered, but these properties are almost necessary.[15]

An asymmetry between a and θ in Equation 6 bears noting: the function $h : \Theta \to \mathbb{R}$ does not have a counterpart function $A \mapsto \mathbb{R}$. The reason is that whether the utility difference between two actions is single crossing or not could be altered by adding a function of a alone to the utility function $u(a, \theta)$. On the other hand, adding a function of θ alone to $u(a, \theta)$ has no such effect because SCD is an ordinal property that is invariant to any (type-dependent) increasing transformation, or *representation*, of u, i.e., $\tilde{u}(a, \theta) \equiv m(u(a, \theta), \theta)$, where each $m(\cdot, \theta) : \mathbb{R} \to \mathbb{R}$ is strictly increasing.

Whether the convexity condition ($\star$) holds can, in general, depend on which representation one chooses. But since SCD is ordinal, the scope of our analysis is in fact broader than it may seem: if a utility function satisfies SCD and some representation satisfies ($\star$), then Theorem 2 applies to that representation.[16] We illustrate how this is useful in Subsection 3.2.3.

If $u(a, \theta)$ has the form (6) with strictly positive functions f_1 and f_2, then up to a positive affine transformation (viz., subtracting $h(\theta)$ and dividing by $f_1(\theta) + f_2(\theta)$),[17] any type's utility becomes a convex combination of two type-independent utility functions over actions, g_1 and g_2. Theorem 2's ratio ordering requirement then simply says that the relative weight on g_1 and g_2 changes monotonically with the agent's type. This idea underlies the following proposition.

Proposition 1. *Let Θ have both a minimum and a maximum (i.e., $\exists\ \underline{\theta}, \overline{\theta} \in \Theta$ such that $(\forall \theta)$ $\underline{\theta} \leq \theta \leq \overline{\theta}$), and the environment (A, Θ, u) be convex. Then, u has SCD* if and only if u has a positive affine transformation $\tilde{u}$ satisfying*

$$\tilde{u}(a, \theta) = \lambda(\theta)\tilde{u}(a, \overline{\theta}) + (1 - \lambda(\theta))\tilde{u}(a, \underline{\theta}), \tag{7}$$

with $\lambda : \Theta \to [0, 1]$ increasing. In addition, if A is minimal, then u has SSCD* if and only if λ is strictly increasing.

Proposition 1 provides an economic interpretation of SCD* as capturing preferences that monotonically shift weight from one extreme type's to the other's. We note, though, that

[15] "Almost" excludes the case in which g_1 and g_2 are affinely dependent, i.e., $g_1 = \lambda g_2 + \gamma$ for some $\lambda, \gamma \in \mathbb{R}$. Intuitively, affine independence ensures that neither g_1 nor g_2 is dispensable in (6).

[16] For example, given a product set $A \subseteq \mathbb{R}^2_+$ and $\theta \in [0, 1]$, the Cobb-Douglas utility $a_1^\theta a_2^{1-\theta}$ does not satisfy ($\star$), but the representation $\theta \log a_1 + (1 - \theta) \log a_2$ is a multidimensional utility (Example 3) and therefore satisfies ($\star$), indeed SCD*.

[17] In general, a positive affine transformation of $u(a, \theta)$ is any $b(\theta)u(a, \theta) + c(\theta)$ where $b(\cdot) > 0$. Unlike arbitrary representations, positive affine transformations preserve condition ($\star$).

even when Θ has extreme types, it could be easier to verify whether a given function u has SCD* using Theorem 2 because one does not have to search among the affine transformations allowed by Proposition 1.

Before turning to the implications of our SCD* characterization for leading examples, we make one last interpretational comment. On the one hand, Theorem 2 and Proposition 1 indicate that SCD—while desirable for tractability (including interval choice), economic intuition, etc.—is a demanding property in a convex environment. On the other hand, because the functions g_1 and g_2 in Theorem 2 are arbitrary, SCD* nevertheless allows for a broad economic landscape. Specifically, when one assumes SCD in any environment (convex or not), one generally has in mind that there is an underlying tradeoff—e.g., risk vs. expected return among lotteries, delay vs. total amount in payment streams, or price vs. product quality in markets—whose balance shifts monotonically with type. Our characterizations show that SCD* is broad enough to capture such desiderata because the g_1 and g_2 functions in Theorem 2 can evaluate the tradeoff differently: e.g., compared to g_2, the function g_1 can be more risk averse, discount the future more, or be more price sensitive; furthermore, as highlighted by Proposition 1, higher types put more weight on one criterion.

3.2. Implications for Leading Examples

This subsection studies the implications of Theorem 2 for our leading examples. For brevity, we focus on the implications of SCD*, stating the SSCD* counterpart only in Corollary 1.

3.2.1. Single-Crossing Expectational Differences

Suppose, following Example 1, that $A \equiv \Delta X$ is a set of lotteries and u is the expected utility induced by $v(x, \theta)$. We say that v has *(strict) single-crossing expectational differences*, or *(S)SCED*, if the expected utility function u has (S)SCD*.[18] SCED is not implied by SCD or even supermodularity of v.[19] Rather:

Corollary 1. *The Bernoulli utility function v has (S)SCED if and only if it has the same form as u in Theorem 2.*

[18] In general, an expected utility function u may not have SSCD* simply because $A \equiv \Delta X$ is not minimal. This arises, for example, when u is a mean-variance utility function and multiple lotteries have the same mean and variance. In such cases we consider utility-indistinguishable lotteries as equivalence classes and the corresponding utility function $\tilde{u}$ defined on these equivalence classes of lotteries. We say v has SSCED if $\tilde{u}$ has SSCD*.

[19] For instance, given $x, \theta \in \mathbb{R}$, any power loss function $v(x, \theta) = -|x - \theta|^z$ is supermodular when $z > 1$, but Corollary 2 below establishes that SCED fails for $z \neq 2$.

The corollary's proof is straightforward: if v satisfies the characterization, then $u(P,\theta) = (\int_x g_1(x)\mathrm{d}P)f_1(\theta) + (\int_x g_2(x)\mathrm{d}P)f_2(\theta) + h(\theta)$ with f_1 and f_2 satisfying the conditions given in Theorem 2, so u has SCD*. Conversely, if $u(P,\theta) \equiv \int_x v(x,\theta)\mathrm{d}P$ has SCD*, and hence has the form in Theorem 2, then so does v because $v(x,\theta) \equiv u(\delta_x,\theta) = g_1(x)f_1(\theta) + g_2(x)f_2(\theta) + h(\theta)$, where δ_x denotes the degenerate lottery on x.

Corollary 1 is related to Abbas and Bell (2011, Theorem 1). They study expected utility over lotteries with some additional restrictions on the environment (e.g., Θ is finite and completely ordered, and preferences satisfy some substantive economic conditions). For that setting, their Theorem 1 states a similar result to the version of Corollary 1 that would obtain using Proposition 1 instead of Theorem 2.[20]

Even aside from the ratio-ordering and single-crossing requirements in Theorem 2, the functional form (6) deserves emphasis: there are only two "interaction terms", each of which is multiplicatively separable in a and θ. This means that preferences over lotteries must be summarized by two linear statistics: for any lottery $P \in \Delta X$, the statistics are $\int_x g_1(x)\mathrm{d}P$ and $\int_x g_2(x)\mathrm{d}P$.[21] This point underlies the following corollary, which identifies quadratic loss as the unique power loss function that has SCED.

Corollary 2. *Let $X = \mathbb{R}$ and $\Theta \subseteq \mathbb{R}$ with $|\Theta| \geq 3$. A loss function $v(x,\theta) = -|x-\theta|^z$ with $z > 0$ has SCED if and only if $z = 2$.*

Under quadratic loss, preferences over lotteries are summarized by the lotteries' first and second moments. We note that some non-power-loss generalizations of quadratic loss, such as $v(x,\theta) = x\theta + g(x) + h(\theta)$ with $g : \mathbb{R} \to \mathbb{R}$, also satisfy SCED; these functional forms—or variants that also augment a quasi-linear money burning component, which continues to preserve SCED—are used in the study of delegation with stochastic mechanisms or money burning (Amador and Bagwell, 2013; Kleiner, 2022).

SCED is useful in dynamic problems, as seen in Banks and Duggan (2006), Duggan (2014), Celik (2015), and Ali, Kartik, and Kleiner (2022). It is thus noteworthy that:

[20] Abbas and Bell (2011) use the terminology of "one-switch independence", which appears equivalent to SSCED up to one minor detail that can be set aside here (concerning the treatment of distinct lotteries that all types are indifferent over). Translated into our notation, they assert that a utility function $v(x,\theta)$ has SSCED if and only if $v(x,\theta) = f_1(\theta)g_1(x) + f_2(\theta)g_2(x) + h(\theta)$, with f_1 strictly positive and f_2/f_1 strictly increasing. This can be seen from our Proposition 1 because, given its hypothesis of extreme types, it implies that v has SSCED if and only if $v(x,\theta) = b(\theta)\left[g_1(x) + \lambda(\theta)(g_2(x) - g_1(x)\right] + c(\theta)$, where b is strictly positive, λ is strictly increasing, and g_1 and g_2 are, respectively, monotonic transformations of $v(\cdot,\overline{\theta})$ and $v(\cdot,\underline{\theta})$.

[21] The utility function $v(x,\theta) = \exp(x\theta)$ cannot be written in the form of Equation 6 and hence does not have SCED. Indeed, its expected utility from an arbitrary lottery cannot be summarized by any finite number of linear statistics, let alone two.

Corollary 3. *Suppose $v(x, \theta)$ has SCED. Then, denoting $x^\infty \equiv (x_t)_{t=0}^\infty$, so does the discounted utility function $\tilde{v}(x^\infty, \theta) = \sum_{t=0}^\infty \rho(t) v(x_t, \theta)$ for any $\rho : \{0, 1, \dots\} \to \mathbb{R}$.*

We omit a proof as the result follows straightforwardly from Corollary 1. Note that Corollary 3 does not require exponential discounting. (Of course, implicitly ρ must ensure that $\tilde{v}(\cdot)$ is finite.)

3.2.2. Single-Crossing Rank-Dependent Expected Utility

Suppose an agent's preferences over lotteries $A \equiv \Delta X$ have a rank-dependent expected utility (RDEU) representation as described in Example 2, with underlying utility $v(x, \theta)$.

Corollary 4. *An RDEU function has SCD* if and only if the underlying utility v has the same form as u in Theorem 2.*

We omit a proof because it is analogous to that for Corollary 1. An RDEU agent thus evaluates a lottery P according to two summary statistics: $\int_x g_1(x) \mathrm{d}(w \circ P)$ and $\int_x g_2(x) \mathrm{d}(w \circ P)$. The difference with expected utility is that these summary statistics are no longer linear in P. Instead the statistics reweight probabilities according to the original reweighting function.

3.2.3. Multidimensional Utility

Suppose, following Example 3, that an agent has a multidimensional utility function $u(a, \theta) \equiv \sum_{i=1}^n g_i(a_i) f_i(\theta)$.

Corollary 5. *A multidimensional utility function u has SCD* if and only if*

$$u(a, \theta) = \left(\sum_{i=1}^n \lambda_i^I g_i(a_i) \right) f^I(\theta) + \left(\sum_{i=1}^n \lambda_i^{II} g_i(a_i) \right) f^{II}(\theta) + h(\theta), \quad (8)$$

with f^I and f^{II} each single crossing and ratio ordered, and $\lambda^I, \lambda^{II} \in \mathbb{R}^n$.

The interpretation is that a multidimensional utility has SCD* when at most two "summary" dimensions matter. The values on these summary dimensions are weighted sums of the original values $g_i(a_i)$ over the primitive dimensions $i = 1, \dots, n$. Higher types place relatively more weight on products/bundles that are more valuable on one of the two summary dimensions; recall Proposition 1.

As mentioned in Example 3, multidimensional utility can capture a consumer choosing among consumption bundles. A canonical specification is the Cobb-Douglas utility

$u(a,\theta) = \prod_{i=1}^{n} a_i^{f_i(\theta)}$, where $a \in \mathbb{R}_+^n$ is the consumption bundle and the vector of $f_i(\theta) \geq 0$ parameterizes the consumer's marginal rates of substitution (MRS). Note that an alternative representation is $\sum_{i=1}^{n} f_i(\theta)\log(a_i)$, which has the multidimensional form. Corollary 5 implies that for Cobb-Douglas utility to have SCD, it must be representable as

$$u(a,\theta) = \left(\prod_{i=1}^{n} a_i^{\lambda_i^I}\right)^{f(\theta)} \left(\prod_{i=1}^{n} a_i^{\lambda_i^{II}}\right)^{1-f(\theta)},$$

with $f : \Theta \to [0,1]$ monotonic and $\lambda^I, \lambda^{II} \geq 0$.[22] The interpretation is that in order to have SCD, the Cobb-Douglas utility must have "two layers": the consumer first evaluates the θ-independent Cobb-Douglas value of two composite goods, and then trades off these composite goods according to a Cobb-Douglas utility function with MRS that is monotonic in θ. SCD guarantees that for any choice set (e.g., a budget set), the value of each composite good in the chosen consumption bundle changes monotonically (necessarily in opposite directions) in θ. A special case is the textbook example of two goods, say 1 and 2, and $u(a,\theta) = a_1^{\theta} a_2^{1-\theta}$ with $\theta \in [0,1]$. In that case, we recover the textbook observation that given any budget set, the consumption of each good is monotonic in the MRS.

As another example, consider discounted utility over consumption streams. For a consumption stream $(c_t)_{t=1}^{T}$, the discounted utility $\sum_{t=1}^{T} c_t \rho(t,\theta)$ is of the multidimensional form. Corollary 5 implies that the discounted utility has SCD* only if, for any consumption stream (c_t), $\sum_t c_t \rho(t,\theta)$ is linearly generated by two functions of θ. By considering consumption streams that are positive only in a single period, we see that each $\rho(t,\cdot)$ must in fact be generated by the same two functions of θ, i.e., $\rho(t,\theta) = \lambda_t^I f^I(\theta) + \lambda_t^{II} f^{II}(\theta)$. When $T \geq 3$, such linear dependency does not hold for exponential discounting, i.e., when each type θ has a discount factor δ_θ such that $\rho(t,\theta) = (\delta_\theta)^t$. Consequently, exponential discounting is incompatible with interval choice: given any three discount factors, there are consumption streams (c_t) and (c'_t) such that an agent with either low or high patience strictly prefers (c_t) while an agent with intermediate patience strictly prefers (c'_t).[23] In fact, our analysis reveals that in general choice between an arbitrary pair of consumption streams will be monotonic in the time preference parameter θ only if $\rho(t,\theta)$ has the aforementioned linear dependence. An example is linear time cost, say $\theta \in [\underline{\theta}, \overline{\theta}] \subset \mathbb{R}$ and $\rho(t,\theta) = \alpha - \theta t$ for some $\alpha > \overline{\theta} T$.[24]

[22] More precisely, Corollary 5 yields the representation $\left(\prod_{i=1}^{n} a_i^{\lambda_i^I}\right)^{f^I(\theta)} \left(\prod_{i=1}^{n} a_i^{\lambda_i^{II}}\right)^{f^{II}(\theta)}$, where f^I and f^{II} are each single crossing and ratio ordered, and Proposition 1 yields the further simplification.

[23] For example, let $(c_t)_{t=1}^3 = (0,5,0)$ and $(c'_t)_{t=1}^3 = (1,0,6)$. The difference $u(c,\theta) - u(c',\theta) = \delta_\theta(1 - 5\delta_\theta + 6\delta_\theta^2)$ is strictly positive if and only if $\delta_\theta \in (1/3, 1/2)$.

[24] The fact that SCD imposes strong restrictions on the nature of discounting can be related to difficulties in

3.2.4. Single-Crossing Expectational Differences over Experiments

Suppose, following Example 4, that Ω, $\Delta\Omega$, and $\Delta\Delta\Omega$ are a set of states, posteriors, and experiments respectively. Given a prior $p^* \in \Delta\Omega$, an agent can only choose a Bayes-plausible experiment: $A \equiv \{Q \in \Delta\Delta\Omega : \int_{p\in\Delta\Omega} p\mathrm{d}Q = p^*\}$. We say that $v(p, \theta)$ has *single-crossing expectational differences over experiments (SCED-X)* if the expected utility function $u(Q, \theta) \equiv \int_p v(p, \theta)\mathrm{d}Q$ over A has SCD*. Since $A \subsetneq \Delta\Delta\Omega$, SCED is sufficient for SCED-X but not necessary. SCED-X is, instead, characterized as follows.

Corollary 6. *For any full-support prior p^*, the function v has SCED-X if and only if*

$$v(p, \theta) = g_1(p)f_1(\theta) + g_2(p)f_2(\theta) + \sum_{\omega} v(\delta_\omega, \theta)p(\omega), \tag{9}$$

with f_1 and f_2 each single crossing and ratio ordered.

As the environment is convex, Theorem 2's characterization applies to the expected utility function u. Similar to the SCED characterization in Corollary 1, the form of the expected utility function u passes to the Bernoulli utility function v.[25] The difference is that the last term in Equation 9 can depend on the posterior p as well as the type θ, unlike the $h(\theta)$ term in Equation 6. This is because the expected utility having the form in Theorem 2 only imposes, for SCED-X, that the expectation of that term is constant over lotteries that average to the prior, rather than over all lotteries. This means that the term can admit a linear dependence in p, as seen in Equation 9.

3.3. Aggregating Single-Crossing Functions

This subsection explains the logic behind Theorem 2; we focus on single crossing here, deferring strict single-crossing to Appendix B.5. Owing to the convex environment, the central issue is when linear aggregations of functions are single crossing. Lemma 1 below shows that ratio ordering is the characterizing property when aggregating two functions; Proposition 2 then establishes that when aggregating more than two functions, no more than two can be linearly independent, which leads to Theorem 2.[26]

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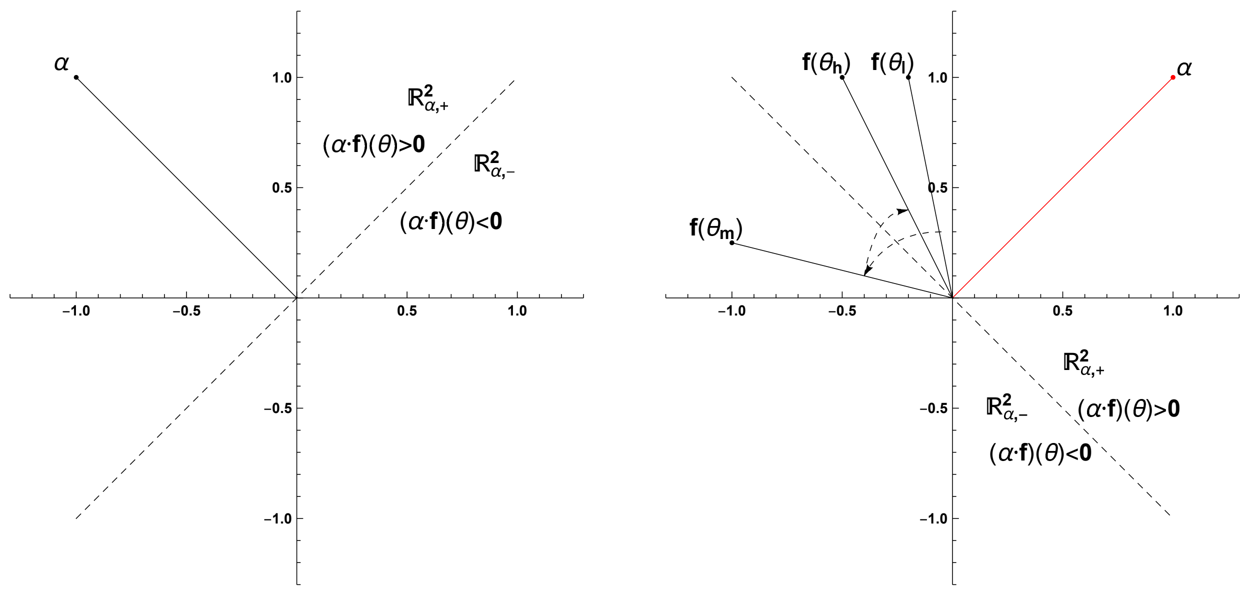
aggregating time preferences (Jackson and Yariv, 2015). Specifically, with three types (or agents), $\underline{\theta} < P < \overline{\theta}$, where "$P$" stands for Planner, SCD is very similar to a Pareto principle: whenever types $\underline{\theta}$ and $\overline{\theta}$ have the same preference over a pair of consumption streams, so should type P. Our discussion indicates that if types $\underline{\theta}$ and $\overline{\theta}$ are exponential discounters, then P cannot be, as was highlighted by Jackson and Yariv (2015). In fact, our result says that P's preference must linearly depend on the other two types, echoing Harsanyi (1955).

[25] This is not as straightforward as in the case SCED, where one can appeal to degenerate lotteries on any posterior. Under SCED-X, the only posterior on which there can be a degenerate lottery is the prior.

[26] Real-valued functions $f_1, \ldots, f_n$ are linearly independent if $(\forall\lambda \in \mathbb{R}^n\backslash\{0\})$ $\sum_{i=1}^n \lambda_i f_i$ is not a zero function, i.e., is not everywhere zero.

Lemma 1. *Let $f_1, f_2 : \Theta \to \mathbb{R}$. The linear combination $\alpha_1 f_1(\theta) + \alpha_2 f_2(\theta)$ is single crossing $\forall \alpha \in \mathbb{R}^2$ if and only if f_1 and f_2 are each single crossing and ratio ordered.*



(a) Sufficiency of ratio ordering.

(b) Necessity of ratio ordering, with $\theta_l < \theta_m < \theta_h$.

Figure 2: Ratio ordering and single crossing of all linear combinations.

Here is the lemma's intuition. For sufficiency, consider any linear combination $\alpha_1 f_1 + \alpha_2 f_2$. Assume $\alpha \in \mathbb{R}^2 \backslash \{0\}$, as otherwise the linear combination is trivially single crossing. The vector α defines two open half spaces $\mathbb{R}^2_{\alpha,-} \equiv \{x \in \mathbb{R}^2 : \alpha \cdot x < 0\}$ and $\mathbb{R}^2_{\alpha,+} \equiv \{x \in \mathbb{R}^2 : \alpha \cdot x > 0\}$, where $\cdot$ is the dot product; see Figure 2(a). As explained earlier after Definition 3, ratio ordering of f_1 and f_2 implies that the vector $f(\theta) \equiv (f_1(\theta), f_2(\theta))$ rotates monotonically as θ increases. If the rotation is from $\mathbb{R}^2_{\alpha,-}$ to $\mathbb{R}^2_{\alpha,+}$ (resp., from $\mathbb{R}^2_{\alpha,+}$ to $\mathbb{R}^2_{\alpha,-}$), then $\alpha \cdot f \equiv \alpha_1 f_1 + \alpha_2 f_2$ is single crossing only from below (resp., only from above). If $\bigcup_{\theta \in \Theta} f(\theta) \subseteq \mathbb{R}^2_{\alpha,-}$ or $\bigcup_{\theta \in \Theta} f(\theta) \subseteq \mathbb{R}^2_{\alpha,+}$, then $\alpha \cdot f$ is single crossing both from below and above. Other cases are similar.[27]

To see why Condition (4) of ratio ordering is necessary, suppose the vector $f(\theta)$ does not rotate monotonically. Figure 2(b) illustrates a case in which, for $\theta_l < \theta_m < \theta_h$, $f(\theta_l)$ rotates counterclockwise to $f(\theta_m)$, but $f(\theta_m)$ rotates clockwise to $f(\theta_h)$. As shown in the Figure, one can find $\alpha \in \mathbb{R}^2$ such that $f(\theta_m) \in \mathbb{R}^2_{\alpha,-}$ while both $f(\theta_l), f(\theta_h) \in \mathbb{R}^2_{\alpha,+}$, which implies that $\alpha \cdot f$ is not single crossing. See Appendix B.1 for why Condition (5) is necessary.

[27] Notice that this argument does not require either f_1 or f_2 to be single-signed. By contrast, Abbas and Bell (2011, p. 769, in their last paragraph on "Necessity") incorrectly claim that all linear combinations of two (strictly) single crossing functions are (strictly) single crossing only if one function is single signed.

Lemma 1 relates to Quah and Strulovici (2012, Proposition 1). They establish that for any two functions f_1 and f_2 that are each single crossing from below, $\alpha_1 f_1 + \alpha_2 f_2$ is single crossing from below for all $\alpha \in \mathbb{R}^2_+$ if and only if f_1 and f_2 satisfy a condition they call signed-ratio monotonicity (see Appendix E for the definition). In general, ratio ordering is not comparable with signed-ratio monotonicity because we consider a different aggregation problem from Quah and Strulovici: (i) the input functions may be single crossing in either direction; (ii) the linear combinations involve coefficients of arbitrary sign; and (iii) the resulting combination can be single crossing in either direction. The example in the introduction highlights the importance of point (ii). There, $\theta \in [-1, 1]$ and $u(a, \theta) = (\theta^2 + 1/2)\mathbb{1}_{\{a=1\}} + 2\mathbb{1}_{\{a=2\}}$. Both $f_1(\theta) = \theta^2 + 1/2$ and $f_2(\theta) = 2$ are positive functions (hence, single crossing from below), and so all positive linear combinations are also positive functions, but $f_1(\theta) - (1/2)f_2(\theta) = \theta^2 - 1/2$ is not single crossing because f_1 and f_2 are not ratio ordered. If the input functions in Lemma 1 are restricted to be single crossing from below, then ratio ordering implies signed-ratio monotonicity.

Lemma 1 implies a characterization of likelihood-ratio ordering for random variables with strictly positive densities. While this likelihood-ratio ordering characterization does not appear to be well-known among economists, it is a special case of Karlin's (1968) results on the variation diminishing property of totally positive functions. More generally, however, Lemma 1 differs from Karlin (1968) because it considers aggregations of functions that can change sign, whereas he only studies non-negative functions.[28]

Theorem 2 requires an extension of Lemma 1 to more than two functions. Consider any set Z and $f : Z \times \Theta \to \mathbb{R}$. We say that f is *linear combinations SC-preserving* if $\int_z f(z, \theta)\mathrm{d}\mu$ is single-crossing in θ for every function $\mu : Z \to \mathbb{R}$ with finite support.[29]

Proposition 2. *Let $f : Z \times \Theta \to \mathbb{R}$ for some set Z. The function f is linear combinations SC-preserving if and only if there exist $z_1, z_2 \in Z$ and $\lambda_1, \lambda_2 : Z \to \mathbb{R}$ such that*

1. *$f(z_1, \cdot) : \Theta \to \mathbb{R}$ and $f(z_2, \cdot) : \Theta \to \mathbb{R}$ are (i) each single crossing and (ii) ratio ordered, and*
2. *$(\forall z)\ f(z, \cdot) = \lambda_1(z) f(z_1, \cdot) + \lambda_2(z) f(z_2, \cdot)$.*

Proposition 2 says that a family of single-crossing functions $\{f(z, \cdot)\}_{z \in Z}$ preserves single crossing of all finite linear combinations if and only if the family is "linearly generated" by two single-crossing functions that are ratio ordered. In particular, given any three single-crossing functions, f_1, f_2, and f_3, all their linear combinations will be single crossing if and

[28] Furthermore, to our knowledge, there is no straightforward approach to transforming the lemma's aggregation problem to only consider non-negative (or single-signed) functions.

[29] The notation $\int_z f(z, \theta)\mathrm{d}\mu$ should be read as $\sum_{\{z:\mu(z)\neq 0\}} f(z, \theta)\mu(z)$.

only if there is a linear dependence in the triple, say $\lambda_1 f_1 + \lambda_2 f_2 = f_3$ for some $\lambda \in \mathbb{R}^2$, and f_1 and f_2 are ratio ordered.

The sufficiency direction of Proposition 2 follows from Lemma 1, as does necessity of the "generating functions" being ratio ordered. The intuition for the necessity of linear dependence is as follows. Assume Θ is completely ordered. For any θ, let $f(\theta) \equiv (f_1(\theta), f_2(\theta), f_3(\theta))$. If $\{f_1, f_2, f_3\}$ is linearly independent, then there exist $\theta_l < \theta_m < \theta_h$ such that $\{f(\theta_l), f(\theta_m), f(\theta_h)\}$ spans $\mathbb{R}^3$. Take any $\alpha \in \mathbb{R}^3 \setminus \{0\}$ that is orthogonal to the plane S_{θ_l,θ_h} that is spanned by $f(\theta_l)$ and $f(\theta_h)$, as illustrated in Figure 3. The linear combination $\alpha \cdot f$ is not single crossing because $(\alpha \cdot f)(\theta_l) = (\alpha \cdot f)(\theta_h) = 0$ while $(\alpha \cdot f)(\theta_m) \neq 0$.

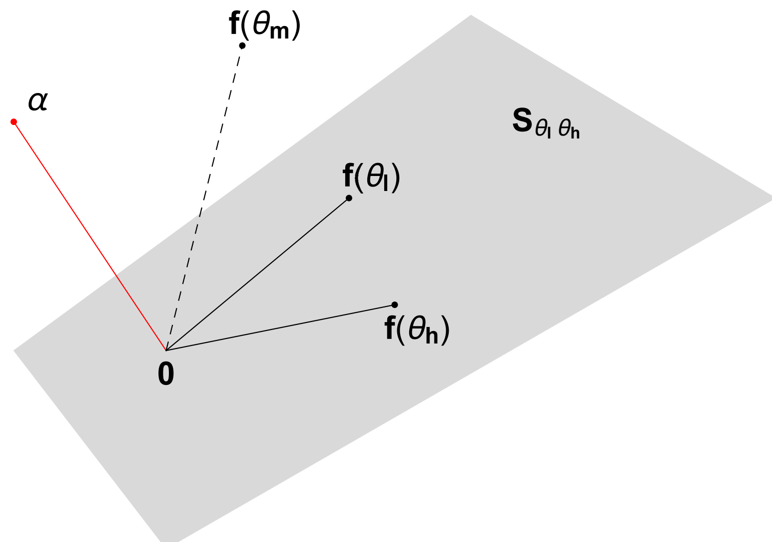


Figure 3: The necessity of linear dependence in Proposition 2.

While the necessity portion of Proposition 2 only asserts ratio ordering of the "generating functions", Lemma 1 implies that if $f : Z \times \Theta \to \mathbb{R}$ is linear combinations SC-preserving, then for all $z, z' \in Z$, the pair $f(z, \cdot) : \Theta \to \mathbb{R}$ and $f(z', \cdot) : \Theta \to \mathbb{R}$ must be ratio ordered.

Proof Sketch of Theorem 2. We can now sketch the argument for Theorem 2. That its characterization is sufficient for SCD* is straightforward from Lemma 1. For necessity, suppose as a simplification that, for some $a_0 \in A$, $(\forall \theta)\ u(a_0, \theta) = 0$.[30] For any $\{a_1, \ldots, a_n\} \subset A$ and $(\lambda_1, \ldots, \lambda_n)$, we build on the Hahn-Jordan decomposition of $(\lambda_1, \ldots, \lambda_n)$ to write the linear combination $\sum_{i=1}^n \lambda_i v(a_i, \theta)$ as $M \sum_{i=0}^n (p(a_i) - q(a_i))u(a_i, \theta)$, where p and q are probability mass functions on $\{a_0, a_1, \ldots, a_n\}$, and M is a scalar.[31] (Unless $\sum_{i=1}^n \lambda_i = 0$,

[30] This is just a normalization, since $u(a, \theta)$ has SCD* if and only if $\tilde{u}(a, \theta) \equiv u(a, \theta) - u(a_0, \theta)$ has SCD*.

[31] Let $L \equiv \sum_{i=1}^n \lambda_i$. For $i > 0$, set $p'(a_i) \equiv \max\{\lambda_i, 0\}$ and $q'(a_i) \equiv -\min\{\lambda_i, 0\}$. If $L \geq 0$, set $p'(a_0) = 0$ and $q'(a_0) \equiv L$; if $L < 0$, set $p'(a_0) \equiv -L$ and $q'(a_0) \equiv 0$. Let $M \equiv \sum_{i=0}^n p'(a_i) = \sum_{i=0}^n q'(a_i)$. Finally, for all

we have $\sum_{i=1}^{n} p(a_i) \neq \sum_{i=1}^{n} q(a_i)$; the assumption that $u(a_0, \cdot) = 0$ permits us to assign all the "excess difference" to a_0, as detailed in fn. 31.) Since the environment is convex there exist $a_p, a_q \in A$ such that the linear combination is equal to $M(u(a_p, \theta) - u(a_q, \theta))$, which is single crossing because u has SCD* . Since every such linear combination is single crossing, Proposition 2 guarantees a' and a'' such that for all a, $u(a, \cdot) = g_1(a)u(a', \cdot) + g_2(a)u(a'', \cdot)$, with $u(a', \cdot)$ and $u(a'', \cdot)$ each single crossing and ratio ordered.

4. Applications

This section illustrates the usefulness of our results in three applications. Interested readers may consult our earlier working paper, Kartik et al. (2019, Section 4.3), for another application to costly signaling.

4.1. Cheap Talk with Uncertain Receiver Preferences

There are two expected-utility players, a sender (S) and a receiver (R). The sender's type is $\theta \in \Theta$, where Θ is ordered by $\leq$. After learning his type, S chooses a payoff-irrelevant message $m \in M$, where $|M| > 1$. After observing m but not θ, R takes a decision $x \in X$. The sender's Bernoulli utility function is $v^S(x, \theta)$; the receiver's is $v^R(x, \theta, \psi)$, where $\psi \in \Psi$ is a preference parameter that is unknown to S when choosing m, and known to R when choosing x. Note that ψ does not affect the sender's preferences. The variables θ and ψ are independently drawn from commonly-known probability distributions.

An example is $\Theta = [0, 1]$, $X = \mathbb{R}$, $\psi \in \Psi \subseteq \mathbb{R}^2$, $v^S(x, \theta) = -(x - \theta)^2$ and $v^R(x, \theta, \psi) = -(x - \psi_1 - \psi_2\theta)^2$. Here the variable ψ_1 captures the receiver's "type-independent bias" and ψ_2 captures the relative "sensitivity" to the sender's type. If ψ were commonly known and θ uniformly distributed, this would be the model of Melumad and Shibano (1991), which itself generalizes the canonical example from Crawford and Sobel (1982) that obtains when $\psi_1 \neq 0$ and $\psi_2 = 1$.

We focus on (weak Perfect Bayesian) equilibria in which S uses a pure strategy, $\mu : \Theta \to M$, and R plays a possibly-mixed strategy, $\alpha : M \times \Psi \to \Delta X$.[32] Given any α, every message m induces some lottery $P_\alpha(m) \in \Delta X$ from the sender's viewpoint. An equilibrium (μ, α) is: (i) an *interval equilibrium* if every message is used by an interval of sender types, i.e., if $(\forall m)$ $\{\theta : \mu(\theta) = m\}$ is an interval; and (ii) *sender minimal* if for all on-the-equilibrium-path $m \neq m'$, there is some θ such that $\mathbb{E}_{P_\alpha(m)}[v^S(\cdot, \theta)] \neq \mathbb{E}_{P_\alpha(m')}[v^S(\cdot, \theta)]$. In other words,

$a \in \{a_0, \dots, a_n\}$, set $p(a) \equiv p'(a)/M$ and $q(a) \equiv q'(a)/M$.

[32] Our notion of equilibrium requires optimal play for every (not just almost every) type of sender. The restriction to pure strategies for the sender is for expositional simplicity.

sender minimality rules out all sender types being indifferent between two distinct on-path messages.[33]

Claim 1. *If v^S has SSCED (Corollary 1) then every sender-minimal equilibrium is an interval equilibrium.*

Proof. From the sender's viewpoint, the lottery over the receiver's decisions that is induced by any message (given any receiver strategy) is independent of θ because ψ and θ are independent. The result follows from Theorem 1, as sender-minimality implies that one can restrict attention to the sender choosing among lotteries that are utility-distinguishable (i.e., if P and Q are equilibrium lotteries, then $D_{P,Q}$ is not a zero function). *Q.E.D.*

Claim 1 relates to Seidmann (1990), who first considered an extension of Crawford and Sobel (1982) to sender uncertainty about the receiver's preferences. His goal was to illustrate how such uncertainty could facilitate informative communication even when the sender always strictly prefers higher receiver decisions. Example 2 in Seidmann (1990) constructs a non-interval and sender-minimal equilibrium that is informative. Claim 1 clarifies that the key is a failure of SSCED.

The strict single crossing property in standard cheap-talk models (e.g., Crawford and Sobel (1982) and Melumad and Shibano (1991)) not only yields interval equilibria, but it also implies that local incentive compatibility is sufficient for global incentive compatibility. This additional tractability also holds under SSCED. Let $\Theta = \{\theta_i : i \in \mathbb{Z}\}$ such that $\theta_i < \theta_j$ for $i < j$, and $P : \Theta \to \Delta X$ be a candidate equilibrium outcome (i.e., the distribution of receiver's choices that each sender type induces in equilibrium). Under SSCED, it is sufficient for sender incentive compatibility that $(\forall i \in \mathbb{Z})\ u(P(\theta_i), \theta_i) \geq \max\{u(P(\theta_{i-1}), \theta_i), u(P(\theta_{i+1}), \theta_i)\}$.

Besides being sufficient, (S)SCED is also necessary to guarantee interval cheap-talk equilibria. Say that v^S *strictly violates* SCED if the expected utility function strictly violates SCD, i.e., if there are $P, Q \in \Delta X$ and $\theta_l < \theta_m < \theta_h$ such that $\min\{D_{P,Q}(\theta_l), D_{P,Q}(\theta_h)\} > 0 > D_{P,Q}(\theta_m)$.

Claim 2. *Let $\Theta \subseteq \mathbb{R}$, $X \equiv \mathbb{R}$, $\Psi \subseteq \mathbb{R}^2$, and $v^R(x, \theta, \psi) \equiv -(x - \psi_1 - \psi_2\theta)^2$. If v^S strictly violates SCED, then for some pair of distributions of θ and ψ there is a non-interval equilibrium in which each sender type plays its unique best response.*

[33] In Crawford and Sobel (1982) and Melumad and Shibano (1991), all equilibria are outcome equivalent to sender-minimal equilibria. More generally, all equilibria are sender minimal when there is a complete order over messages under which higher messages are infinitesimally more costly for all sender types.

Proof. Assume v^S strictly violates SCED and let P and Q be the distributions and $\theta_l < \theta_m < \theta_h$ the types in that definition. In what follows, we can without loss assume $|M| = 2$. So let $M \equiv \{m', m''\}$ and consider the sender's strategy

$$\mu(\theta) = \begin{cases} m' & \text{if } \theta \in \{\theta_l, \theta_h\} \\ m'' & \text{if } \theta = \theta_m. \end{cases}$$

Let F_θ be any distribution with support $\{\theta_l, \theta_m, \theta_h\}$ and $\theta' \equiv \mathbb{E}_{F_\theta}[\theta|\theta \in \{\theta_l, \theta_h\}] \neq \theta_m$. Then, the unique best response against μ for a receiver of type $\psi = (\psi_1, \psi_2)$ is

$$\alpha(m', \psi) = \psi_1 + \psi_2\theta' \quad \text{and} \quad \alpha(m'', \psi) = \psi_1 + \psi_2\theta_m.$$

It can be verified that there is a distribution F_ψ such that, from the sender's viewpoint, the message m' leads to the distribution P and the message m'' leads to the distribution Q, and so μ is the sender's unique best response.[34] *Q.E.D.*

The particular specification of v^R in Claim 2 is not critical; what is important is that there be enough flexibility to generate appropriate lotteries from the sender's viewpoint using best responses for the receiver.[35] For example, the result would also hold—more straightforwardly, but less interestingly—if the receiver were totally indifferent over all actions for some preference realization. On the other hand, if $\psi \in \mathbb{R}$ and $v^R(x, \theta, \psi) \equiv -(x - \theta - \psi)^2$, then (S)SCED is not necessary, because any pair of lotteries that the sender may face are ranked by first-order stochastic dominance. Strict supermodularity of $v^S(x, \theta)$ then guarantees that all sender-minimal equilibria are interval equilibria; however, strict supermodularity does not imply (S)SCED, as noted in Subsection 3.2.1.

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[34] Let x and y be random variables with distributions P and Q, respectively. Let F_ψ be the distribution of a random vector $\psi = (\psi_1, \psi_2)$ defined by

$$\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \equiv \begin{bmatrix} 1 & \theta' \\ 1 & \theta_m \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix}.$$

As $\psi \sim F_\psi$ (i.e., ψ has distribution F_ψ),

$$\begin{bmatrix} 1 & \theta' \\ 1 & \theta_m \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} \psi_1 + \psi_2\theta' \\ \psi_1 + \psi_2\theta_m \end{bmatrix}$$

is stochastically equivalent to (x, y). Thus, $\alpha(m', \psi) = \psi_1 + \psi_2\theta' \sim P$ and $\alpha(m'', \psi) = \psi_1 + \psi_2\theta_m \sim Q$.

[35] In particular, the result in Claim 2 holds under the following more general assumptions: $\Theta, A \subseteq \mathbb{R}$, the receiver's preferences are represented by $u^R(a, \theta, \psi) = g_1(\psi, a)\theta + g_2(\psi, a)$, and for every $\theta_l < \theta_h$ and $a', a'' \in A$, there exists $\psi \in \Psi$ such that $a' \in \arg\max_a u^R(a, \psi, \theta_l)$ and $a'' \in \arg\max_a u^R(a, \psi, \theta_h)$. The proof is very similar to that of Claim 2.

In our cheap-talk application it is uncertainty about the receiver's preferences that leads to the sender effectively choosing among lotteries over the receiver's decisions. Similar results could also be obtained when the receiver's preferences are known but communication is noisy, à la Blume, Board, and Kawamura (2007).

4.2. Observational Learning with Multidimensional Utility

The classic sequential observational learning model (Banerjee, 1992; Bikhchandani, Hirshleifer, and Welch, 1992) considers agents sequentially choosing products with uncertainty about product values but learning from predecessors' choices.

For concreteness, suppose that students $t = 1, 2, \ldots$ sequentially purchase laptops before entering an engineering school. Each laptop is identified by its attribute vector $a \equiv (a_1, \ldots, a_n) \in [0,1]^n$, which consists of processing power, memory, screen size, price, etc. The students are uncertain about the nature of the work required to complete the degree, which is captured by a state $\theta \in \Theta \subset \mathbb{R}$, where Θ is countable. The state affects how students weigh each attribute of a laptop. Specifically, following Example 3, students have a common multidimensional utility function $u(a, \theta) \equiv \sum_{i=1}^{n} g_i(a_i) f_i(\theta)$. Each g_i has a convex range, which ensures a convex environment, i.e., property ($\star$). Students are expected-utility maximizers.

Given a finite set of available laptops $\mathcal{A} \subseteq [0,1]^n$ and a prior μ_0 over the state, each student chooses a laptop using two information sources. Conditional on the state, student t obtains an independent private signal s_t about θ (e.g., her own impression from reading syllabi and talking to alumni). A canonical example is *normal information*: $s_t \sim \mathcal{N}(\theta, \sigma^2)$, where $\sigma > 0$ is a known constant. Each student also observes all predecessors' choices.

The question is whether there is *adequate learning*: no matter the prior μ_0 and the choice set $\mathcal{A}$, do students' sequential choices eventually settle on the laptop they would choose if the state were known? See Kartik, Lee, Liu, and Rappoport (2022) for a precise definition. Those authors identify SCD as the necessary and sufficient condition on the utility function $u(a, \theta)$ for learning under normal information, and more generally, for information structures that have *directionally unbounded beliefs*.[36] In the current multidimensional utility environment, the following characterization obtains, which we state without proof.

Claim 3. *There is adequate learning under normal information if and only if $u(a, \theta)$ has SCD*, i.e., it has the form stated in Corollary 5.*

[36] Roughly, an information structure has directionally unbounded beliefs if for every state there exist signals that provide arbitrarily-close-to relative certainty about that state vs. all lower states, and analogously (potentially different) signals for that state vs. all higher states.

The claim implies that adequate learning under normal information obtains if only if students' preferences depend on at most two "summary attributes", which are each linear combinations over the original value over attributes, i.e., linear combinations of the functions $g_i(a_i)$. For example, students may trade off "convenience", which is an aggregate of price, weight, and size, with "performance", which aggregates processing speed, memory, and storage. Learning requires that the convenience and performance aggregators do not vary with the state. A second implication derives from f^I and f^{II} in Equation 8 being ratio ordered. The interpretation is that for learning, students must value one summary attribute more as the state increases, e.g., a higher state indicates more computationally intensive work and hence a higher value of performance relative to convenience.

4.3. Collective Choice over Lotteries

Collective choice over lotteries arises naturally in many contexts: elections entail uncertainty about what policies politicians will implement if elected; and a board of directors may view each candidate for CEO as a probability distribution over firm earnings. Zeckhauser (1969) first pointed out that pairwise-majority comparisons in these settings can be cyclical, even when comparisons over deterministic outcomes are not. Our results shed light on when such difficulties do not arise.

Consider a finite group of individuals indexed by $i \in \{1, 2, \ldots, N\}$, where for simplicity N is odd. The group must choose from a set of lotteries over X, where X is the space of outcomes (political policies, earnings, etc.) with generic element x. Each individual i has Bernoulli utility function $v(x, \theta_i)$, where $\theta_i \in \Theta$ is a preference parameter or i's type. We assume Θ is completely ordered; without further loss of generality, let $\Theta \subset \mathbb{R}$ and $\theta_1 \leq \cdots \leq \theta_N$. The expected utility for an individual of type θ from lottery P is $u(P, \theta) \equiv \int_x v(x, \theta) \mathrm{d}P$. We denote individual i's preference relation over lotteries by $\succeq_i$, with strict component $\succ_i$.

Define the group's preference relation, $\succeq_{maj}$, over lotteries P and Q by majority rule:

$$P \succeq_{maj} Q \text{ if } |\{i : P \succeq_i Q\}| \geq N/2.$$

Claim 4. *If v has SCED (Corollary 1), then the group's preference relation is transitive and coincides with that of individual $(N + 1)/2$.*

Our proof is related to the monotone comparative statics arguments of Gans and Smart (1996), but highlights that an order on the choice set is unnecessary.

Proof. Let $M \equiv (N + 1)/2$. By SCD, (i) if $P \succeq_M Q$, then $P \succeq_{maj} Q$ because there cannot exist $i < M < j$ such that $Q \succ_i P$ and $Q \succ_j P$; and, analogously, (ii) if $P \succ_M Q$, then $P \succ_{maj} Q$. Hence, $\succeq_{maj}$ coincides with $\succeq_M$. *Q.E.D.*

Claim 4 can be applied to a well-known problem in political economy (Shepsle, 1972). The policy space is a finite set $X \subset \mathbb{R}$ (for simplicity) and there are an odd N number of voters ordered by their ideal points in $\mathbb{R}$, $\theta_1 \leq \cdots \leq \theta_N$ (i.e., for each voter i, $\{\theta_i\} = \arg\max_{x \in \mathbb{R}} v(x, \theta_i)$). Let $M \equiv (N+1)/2$. There are two office-motivated candidates, L and R; each $j \in \{L, R\}$ can commit to any lottery from some given set $\mathcal{A}_j \subseteq \Delta X$. A restricted set $\mathcal{A}_j$ may capture various kinds of constraints; for example, Shepsle (1972) assumed the incumbent candidate could only choose degenerate lotteries. In our setting, what ensures the existence of an equilibrium, and which policy lotteries are offered in an equilibrium?[37]

Claim 4 implies that if voters' utility functions v have SCED and if voter M is indifferent between her most-preferred lottery in $\mathcal{A}_L$ and in $\mathcal{A}_R$, then there is a unique equilibrium: each candidate offers the best lottery for voter M; in particular, both candidates converge to δ_{θ_M}, the degenerate lottery on θ_M, if that is feasible for both. It bears emphasis, however, that in this case policy convergence at the median ideal point is not driven by all voters being globally "risk averse" (e.g., $v(x, \theta) = -(x - \theta)^2$); rather, it is because SCED ensures the existence of a decisive voter whose most-preferred lottery is degenerate.[38]

There is a sense in which SCED is necessary to guarantee that each candidate j will offer the median ideal-point voter's most-preferred lottery from the feasible set $\mathcal{A}_j$. Suppose $v(x, \theta)$ strictly violates SCED, i.e., there are $P, Q \in \Delta X$ and $\theta_l < \theta_m < \theta_h$ such that $\min\{D_{P,Q}(\theta_l), D_{P,Q}(\theta_h)\} > 0 > D_{P,Q}(\theta_m)$. Then, if the population of voters is just $\{l, m, h\}$ and $\mathcal{A}_L = \mathcal{A}_R = \{P, Q\}$, the unique equilibrium is for both candidates to offer lottery P, which is voter m's less preferred lottery.

5. Discussion

5.1. Single Crossing vs. Monotonicity

We have characterized when $u : A \times \Theta \to \mathbb{R}$ has SCD in a convex environment. SCD is an ordinal property. Analogous to the interest in monotonic functions rather than just

[37] More precisely: the two candidates simultaneously choose their lotteries, and each voter then votes for his preferred candidate (assuming, for concreteness, that a voter randomizes between the candidates with equal probability if indifferent). A candidate wins if he receives a majority of the votes. Candidates maximize the probability of winning. We seek a Nash equilibrium of the game between the two candidates.

[38] An example may be helpful. Let $X = [-1, 1]$, $\Theta = \{-1, 0, 1\}$, and $v(x, \theta) = x\theta + 1/(|x| + 1) + 1$. The corresponding functions $f_1(\theta) = \theta$ and $f_2(\theta) = 1$ are each strictly single crossing from below and strictly ratio ordered. For all θ, $v(\cdot, \theta)$ is maximized at $x = \theta$ but is convex on some subinterval of X.

single-crossing functions, one may also be interested in the following cardinal property, which we term *monotonic differences* (MD):

$$(\forall a, a' \in A)\ \ u(a,\theta) - u(a',\theta) \text{ is monotonic in } \theta.$$

For a convex environment, we write MD* instead of MD.

Theorem 3. *The function $u : A \times \Theta \to \mathbb{R}$ has MD* if and only if it takes the form*

$$u(a,\theta) = g_1(a) f_1(\theta) + g_2(a) + h(\theta), \tag{10}$$

where f_1 is monotonic.

Suppose that an expected-utility agent with Bernoulli utility $v(x,\theta)$ is choosing among lotteries, so $A \equiv \Delta X$. We say that v has *monotonic expectational differences* (MED) if the expected utility function u has MD*. Analogous to the SCED characterization in Corollary 1, it is straightforward that the function v has MED if and only if it has the same characterization as given for u in Theorem 3.[39] This MED characterization has largely been obtained by Kushnir and Liu (2018) in their study of the equivalence between Bayesian and dominant-strategy implementation.[40]

A function u with SCD* has MD* when the function f_2 in the form (6) is identically equal to one; f_1 and f_2 being ratio ordered is then equivalent to f_1 being monotonic. In general, SCD* functions need not take the MD* form. However, by Proposition 1, SCD* functions have MD* *representations* under a reasonable condition on the environment.[41] We view this finding as unexpected; outside of convex environments, we are not aware of any general result on when SCD functions have MD representations.

[39] There is actually a simple intuition for the MED characterization based on the von Neumann-Morgenstern expected utility theorem. Suppose $\Theta = [\underline{\theta}, \overline{\theta}] \subset \mathbb{R}$ and $v_\theta(a,\theta)$, the partial derivative of v with respect to θ, exists and is continuous. Consider the following strengthening of MED: $(\forall P, Q \in \Delta X)$ $D_{P,Q}(\theta)$ is either a zero function or strictly monotonic. Then, for any P and Q, sign $\int_x v_\theta(x,\theta)\mathrm{d}[P-Q]$ is independent of θ. In other words, for all θ, $v_\theta(\cdot,\theta)$ is a von Neumann-Morgenstern representation of the same preferences over lotteries. The MED characterization follows from the expected utility theorem's implication that $(\forall \theta', \theta'')$ $v_\theta(\cdot,\theta')$ must be a positive affine transformation of $v_\theta(\cdot,\theta'')$.

[40] More precisely, they characterize a slightly stronger property than MED, called "increasing differences over distributions", given some additional assumptions on the environment. They restrict attention to $\Theta \subseteq \mathbb{R}$, $X \subset \mathbb{R}^k$, and functions v that have some continuity.

[41] The condition is that Θ has both a minimum and a maximum. Without this condition, an SCD* function may not have an MD* representation. In the context of expected utility, an earlier version of our paper characterized precisely when such representations do not exist and provided an example (Kartik et al., 2019, Appendix G). The difficulties of finding SCED functions that do not have MED representations can be connected to the concluding discussion in Duggan (2014, Section 4).

5.2. Interval Choice and Monotone Comparative Statics

In Section 2, we showed that (S)SCD characterizes interval choice. There is a sense in which interval choice is intimately related to monotone comparative statics holding with respect to *some* order over the choice space. In light of Theorem 1, the connection is elucidated below by tying (S)SCD to monotone comparative statics.

Throughout this subsection, we consider an ordered set of alternatives, $(A, \succeq)$. To simplify exposition and avoid dealing with equivalence classes, we will focus on comparative statics for a function $u : A \times \Theta \to \mathbb{R}$ such that A is minimal (with respect to u), i.e., $(\forall a \neq a')(\exists \theta)\, u(a, \theta) \neq u(a', \theta)$. For any $a, a' \in A$, let $a \vee a'$ and $a \wedge a'$ denote the usual join and meet respectively.[42] Neither need exist. Given any $S, S' \subseteq A$, we say that S dominates S' in the *strong set order*, denoted $S \succeq_{SSO} S'$, if for every $a \in S$ and $a' \in S'$, (i) $a \vee a'$ and $a \wedge a'$ exist, and (ii) $a \vee a' \in S$ and $a \wedge a' \in S'$. The strong set order is transitive on non-empty subsets of $(A, \succeq)$. While this transitivity is well-known when $(A, \succeq)$ is a lattice, it can be shown to be a general property.

Definition 4. $u : A \times \Theta \to \mathbb{R}$ has *monotone comparative statics (MCS)* on $(A, \succeq)$ if

$$(\forall S \subseteq A)\ (\forall \theta_l \leq \theta_h) \quad \arg\max_{a \in S} u(a, \theta_h) \succeq_{SSO} \arg\max_{a \in S} u(a, \theta_l).$$

Our definition of MCS is closely related to but not the same as Milgrom and Shannon (1994). We take $(A, \succeq)$ to be any ordered set while they require a lattice. We focus only on monotonicity of choice in θ but require the monotonicity to hold for every subset $S \subseteq A$; Milgrom and Shannon require monotonicity of choice jointly in the pair (θ, S), but this implicitly only requires choice monotonicity in θ to hold for every sublattice $S \subseteq A$.

Define binary relations $\succ_{SCD}$ and $\succeq_{SCD}$ on A as follows: $a \succ_{SCD} a'$ if $D_{a,a'}(\theta) \equiv u(a, \theta) - u(a', \theta)$ is single crossing *only* from below; $a \succeq_{SCD} a'$ if either $a \succ_{SCD} a'$ or $a = a'$. It is clear that $\succeq_{SCD}$ is reflexive and anti-symmetric. If $u : A \times \Theta \to \mathbb{R}$ has SCD, then $\succeq_{SCD}$ is also transitive.[43] Moreover, given SCD (and minimality), the $\succeq_{SCD}$ order is incomplete only over pairs with "dominance": if $a \not\succeq_{SCD} a'$ and vice-versa, then either $(\forall \theta)\ D_{a,a'}(\theta) > 0$, or $(\forall \theta)\ D_{a,a'}(\theta) < 0$.

[42] $\overline{a} \in A$ is the the join (or supremum) of $\{a, a'\}$ if (i) $\overline{a} \succeq a$ and $\overline{a} \succeq a'$, and (ii) if $b \succeq a$ and $b \succeq a'$, then $b \succeq \overline{a}$. The meet (or infimum) of $\{a, a'\}$ is defined analogously.

[43] To confirm transitivity, take $a, a', a'' \in A$ such that $a \succeq_{SCD} a'$ and $a' \succeq_{SCD} a''$. If $a = a'$ or $a' = a''$, it is trivial that $a \succeq_{SCD} a''$. If $a \neq a'$ and $a' \neq a''$, then $a \succ_{SCD} a'$ and $a' \succ_{SCD} a''$. Since u has SCD, $D_{a,a''}$ is single crossing. As both $D_{a,a'}$ and $D_{a',a''}$ are single crossing only from below, $D_{a,a''} = D_{a,a'} + D_{a',a''}$ is single crossing only from below. Thus, $a \succ_{SCD} a''$.

Theorem 4. *$u : A \times \Theta \to \mathbb{R}$ has monotone comparative statics on $(A, \succeq)$, where A is minimal, if and only if u has SCD and $\succeq$ is a refinement of $\succeq_{SCD}$.*[44]

Our definition of SCD does not require an order on the set of alternatives, whereas MCS does. Theorem 4 says that SCD is necessary and sufficient for there to exist an order that yields MCS. Moreover, the Theorem justifies viewing $\succeq_{SCD}$ as the prominent order for MCS: MCS does not hold with any order that either coarsens $\succeq_{SCD}$ or reverses a ranking by $\succ_{SCD}$. The argument for necessity in Theorem 4 only makes use of binary choice sets. If SCD fails, then there is no order $\succeq$ for which there is choice monotonicity for all binary choice sets. If SCD holds, then choice monotonicity on all binary choice sets requires $\succeq$ to refine $\succeq_{SCD}$.

Regarding sufficiency, for each $S \subseteq A$, let $C_u(S) \equiv \bigcup_{\theta \in \Theta} \arg\max_{a \in S} u(a, \theta)$. Given that u has SCD and A is minimal, the set $C_u(S)$ is completely ordered by $\succeq_{SCD}$ (as elaborated in the proof of Theorem 4). Since $\succeq$ is a refinement of $\succeq_{SCD}$, $\succeq$ coincides with $\succeq_{SCD}$ on $C_u(S)$, and the strong set orders generated by $\succeq$ and $\succeq_{SCD}$ on the collection of all subsets of $C_u(S)$ also coincide. By definition of $\succeq_{SCD}$, u satisfies Milgrom and Shannon's (1994) single-crossing property in (a, θ) with respect to $\succeq_{SCD}$ and $\leq$. It follows from Milgrom and Shannon (1994, Theorem 4) that $\forall \theta_l \leq \theta_h$,

$$\arg\max_{a \in S} u(a, \theta_h) = \arg\max_{a \in C_u(S)} u(a, \theta_h) \succeq_{SSO} \arg\max_{a \in C_u(S)} u(a, \theta_l) = \arg\max_{a \in S} u(a, \theta_l).$$

Notice that, unlike in Milgrom and Shannon (1994, Theorem 4), quasisupermodularity does not appear in Theorem 4.[45] Regarding necessity, the reason is that, as noted earlier, we do not require monotonicity of choice as the choice set varies. Regarding sufficiency, the reason is that our construction of the order $\succeq_{SCD}$ is complete "enough", i.e., for any $S \subseteq A$ it is complete over $C_u(S)$ defined in the previous paragraph.

A stronger notion of choice monotonicity is given by the next definition.

Definition 5. $u : A \times \Theta \to \mathbb{R}$ has *monotone selection (MS)* on $(A, \succeq)$ if for any $S \subseteq A$, every selection $s^*(\theta)$ from $\arg\max_{a \in S} u(a, \theta)$ is increasing in θ.[46]

Define binary relations $\succ_{SSCD}$ and $\succeq_{SSCD}$ on A as follows: $a \succ_{SSCD} a'$ if $D_{a,a'}$ is strictly single crossing only from below; $a \succeq_{SSCD} a'$ if either $a \succ_{SSCD} a'$ or $a = a'$. As before, if $u : A \times \Theta \to \mathbb{R}$ has SSCD, then $\succeq_{SSCD}$ is an order.

[44] Given two orders $\succeq$ and $\succeq'$ on A, the order $\succeq$ is a refinement of $\succeq'$ if $(\forall a, a' \in A)\ a \succeq' a' \implies a \succeq a'$.

[45] On a lattice $(Z, \succeq)$, $h : Z \to \mathbb{R}$ is quasisupermodular if $h(z) \geq (>) h(z \wedge z') \implies h(z \vee z') \geq (>) h(z')$.

[46] $s^*(\theta) \equiv \emptyset$ if $\arg\max_{a \in S} u(a, \theta) = \emptyset$, and we extend $\succeq$ to $A \cup \{\emptyset\}$ by stipulating $a \succeq \emptyset \succeq a$ for every $a \in A$.

Theorem 5. *$u : A \times \Theta \to \mathbb{R}$ has monotone selection on $(A, \succeq)$, where A is minimal, if and only if u has SSCD and $\succeq$ is a refinement of $\succeq_{SSCD}$.*

Note that SSCD (and a refinement of $\succeq_{SSCD}$) is not only sufficient but also necessary in Theorem 5. One can verify that, while not stated in their Theorem 4′, Milgrom and Shannon's (1994) strict single crossing property is also necessary for monotone selection in their sense.

6. Conclusion

This paper has characterized the class of utility functions that have SCD*, i.e., single-crossing differences in a convex environment (Theorem 2). Our notion of SCD does not presume an order on the choice space and is the appropriate notion for interval-choice comparative statics (Theorem 1). We have given a number of examples of convex environments, most notably expected utility, rank-dependent expected utility, and multidimensional utility, and discussed the implications of our characterization in these contexts. The applications in Section 4 illustrate how SCD* is useful in economic problems.

We close by mentioning some avenues for future research. First, it may be of interest to characterize exactly when preferences have a utility representation that satisfies our convexity condition ($\star$). Second, and relatedly, one may explore characterizations of SCD outside convex environments. In particular, we are intrigued by the question of when SCD preferences do *not* possess a representation that takes a similar form to that we have characterized.

Third, our results have direct bearing on problems in which all types of an agent face the same choice set. Such situations are natural. But consider a variation of the cheap-talk application (Subsection 4.1) in which the sender's type is correlated with the receiver's type. Even though the receiver's type does not affect the sender's payoff, different sender types will generally have different beliefs about the distribution of the receiver's action that any message induces in equilibrium. Effectively, different sender types will be choosing from different choice sets. An approach that synthesizes the current paper's with that of, for example, Athey's (2002) may be useful for such problems.

Appendices

Organization. Appendix A contains proofs for our comparative statics results (Theorem 1, Theorem 4, and Theorem 5); Appendix B for aggregation of single crossing functions (Lemma 1 and Proposition 2) and our characterization of (S)SCD* (Theorem 2 and Proposition 1); Appendix C for the implications of Theorem 2 (Corollary 1, Corollary 2, Corollary 5, and Corollary 6); and Appendix D for our MD* characterization (Theorem 3). Finally, Appendix E elaborates on the connection between ratio ordering and Quah and Strulovici's (2012) signed-ratio monotonicity.

A preliminary result. Before turning to the proofs, we state a useful equivalence with the violation of single crossing; the result is obvious when $(\Theta, \leq)$ is a completely ordered set but also applies when it is not.

Claim 5. *A function $f : \Theta \to \mathbb{R}$ is not single crossing if and only if for some $\theta_l < \theta_m < \theta_h$, either*

$$\operatorname{sign}[f(\theta_l)] < \operatorname{sign}[f(\theta_m)] \textit{ and } \operatorname{sign}[f(\theta_m)] > \operatorname{sign}[f(\theta_h)], \quad \textit{or} \tag{11}$$

$$\operatorname{sign}[f(\theta_l)] > \operatorname{sign}[f(\theta_m)] \textit{ and } \operatorname{sign}[f(\theta_m)] < \operatorname{sign}[f(\theta_h)]. \tag{12}$$

Proof of Claim 5. The "if" direction of the claim is immediate. For the "only if" direction, suppose $f : \Theta \to \mathbb{R}$ is single crossing neither from below nor from above:

$$(\exists \theta_1 < \theta_2) \quad \operatorname{sign}[f(\theta_1)] < \operatorname{sign}[f(\theta_2)], \text{ and}$$
$$(\exists \theta_3 < \theta_4) \quad \operatorname{sign}[f(\theta_3)] > \operatorname{sign}[f(\theta_4)].$$

Let $\Theta_0 \equiv \{\theta_1, \theta_2, \theta_3, \theta_4\}$ and $\overline{\theta}$ and $\underline{\theta}$ be upper and lower bounds of Θ_0. If $f(\underline{\theta}) = f(\overline{\theta}) = 0$, then $(\theta_l, \theta_m, \theta_h) = (\underline{\theta}, \theta_0, \overline{\theta})$ for some $\theta_0 \in \Theta_0$ with $f(\theta_0) \neq 0$ satisfies either (11) or (12). So assume $f(\overline{\theta}) \neq 0$; an similar argument applies if $f(\underline{\theta}) \neq 0$. If $f(\overline{\theta}) < 0$, then $(\theta_l, \theta_m, \theta_h) = (\theta_1, \theta_2, \overline{\theta})$ satisfies (11). If $f(\overline{\theta}) > 0$, then $(\theta_l, \theta_m, \theta_h) = (\theta_3, \theta_4, \overline{\theta})$ satisfies (12). *Q.E.D.*

A. Proofs for Comparative Statics

A.1. Proof of Theorem 1

Part 1. Suppose u has SCD, and consider any $S \subseteq A$, $a' \in S$, and $\theta_l, \theta_h \in \{\theta : a' \in \arg\max_{a \in S} u(a, \theta)\}$ with $\theta_l < \theta_h$. For any $a'' \in S$, $D_{a',a''}(\theta_l) \geq 0$ and $D_{a',a''}(\theta_h) \geq 0$, which imply that $D_{a',a''}(\theta_m) \geq 0$ for all θ_m with $\theta_l < \theta_m < \theta_h$. It follows that $\{\theta : a' \in \arg\max_{a \in S} u(a, \theta)\}$ is an interval.

If u strictly violates SCD, $\exists a', a'' \in A$ and $\theta_l < \theta_m < \theta_h$ such that $\min\{D_{a',a''}(\theta_l), D_{a',a''}(\theta_h)\} > 0 > D_{a',a''}(\theta_m)$. Clearly, $\{\theta : a' \in \arg\max_{a \in \{a',a''\}} u(a,\theta)\}$ is not an interval.

Part 2. Suppose $|\Theta| \geq 3$. A function $u : A \times \Theta \to \mathbb{R}$ does not have SSCD when there exist $a', a'' \in A$ such that $D_{a',a''}$ (and $D_{a'',a'}$) are not strictly single crossing. Alternatively, using Claim 5, u does not have SSCD if and only if $(\exists a', a'')$ $(\exists \theta_l < \theta_m < \theta_h)$ $D_{a',a''}(\theta_l) \geq 0$, $D_{a',a''}(\theta_m) \leq 0$, and $D_{a',a''}(\theta_h) \geq 0$. This condition is equivalent to $(\exists S \subseteq A$ with $a', a'' \in S)$ $(\exists \theta_l < \theta_m < \theta_h)$ $a' \in \bigcap_{\theta \in \{\theta_l, \theta\}} \arg\max_{a \in S} u(a,\theta)$ and $D_{a',a''}(\theta_m) \leq 0$, which holds if and only if some selection from the choice correspondence C_u does not have interval choice.

A.2. Proof of Theorem 4

$(\implies)$ Suppose $u : A \times \Theta \to \mathbb{R}$ has MCS on A with some order $\succeq$. We first prove the following claim.

Claim 6. *For every $a', a'' \in A$, if $\exists \theta_l < \theta_h$ such that $\operatorname{sign}[D_{a',a''}(\theta_l)] < \operatorname{sign}[D_{a',a''}(\theta_h)]$, then $a' \succ a''$.*

Proof. Consider $S = \{a', a''\}$. Since $\operatorname{sign}[D_{a',a''}(\theta_l)] \neq \operatorname{sign}[D_{a',a''}(\theta_h)]$, we have

$$\arg\max_{a \in S} u(a, \theta_l) \neq \arg\max_{a \in S} u(a, \theta_h).$$

Thus, either (i) $a' \in \arg\max_{a \in S} u(a,\theta_l)$ and $a'' \in \arg\max_{a \in S} u(a,\theta_h)$, or (ii) $a'' \in \arg\max_{a \in S} u(a,\theta_l)$ and $a' \in \arg\max_{a \in S} u(a,\theta_h)$. Since u has MCS on $(A, \succeq)$, we have $\arg\max_{a \in S} u(a,\theta_h) \succeq_{SSO} \arg\max_{a \in S} u(a,\theta_l)$. Therefore, $a' \wedge a'' \in \arg\max_{a \in S} u(a,\theta_l)$ and $a' \vee a'' \in \arg\max_{a \in S} u(a,\theta_h)$, which implies that either $a' \succeq a''$ or $a'' \succeq a'$. Since $a' \neq a''$, we have either $a' \succ a''$ or $a'' \succ a'$. If $a'' \succ a'$, then $a'' = a' \vee a'' \in \arg\max_{a \in S} u(a,\theta_h)$, contradicting $\operatorname{sign}[D_{a',a''}(\theta_l)] < \operatorname{sign}[D_{a',a''}(\theta_h)]$. Thus, $a' \succ a''$. *Q.E.D.*

To show that u has SCD on A, suppose not, per contra. Claim 5 implies there exist $a', a'' \in A$ and $\theta_l < \theta_m < \theta_h$ such that either

$$\operatorname{sign}[D_{a',a''}(\theta_l)] < \operatorname{sign}[D_{a',a''}(\theta_m)] \quad \text{and} \quad \operatorname{sign}[D_{a',a''}(\theta_m)] > \operatorname{sign}[D_{a',a''}(\theta_h)], \quad \text{or} \tag{13}$$

$$\operatorname{sign}[D_{a',a''}(\theta_l)] > \operatorname{sign}[D_{a',a''}(\theta_m)] \quad \text{and} \quad \operatorname{sign}[D_{a',a''}(\theta_m)] < \operatorname{sign}[D_{a',a''}(\theta_h)]. \tag{14}$$

Given either (13) or (14), Claim 6 implies $a' \succ a''$ and $a'' \succ a'$, a contradiction.

To show that $\succeq$ is a refinement of $\succeq_{SCD}$, it suffices to show that

$$(\forall a', a'' \in A) \quad a' \succ_{SCD} a'' \implies a' \succ a'', \tag{15}$$

because both $\succeq$ and $\succeq_{SCD}$ are anti-symmetric. Take any $a', a'' \in A$ such that $a' \succ_{SCD} a''$. As $D_{a',a''}$ is single crossing only from below, $\exists \theta_l < \theta_h$ such that $\text{sign}[D_{a',a''}(\theta_l)] < \text{sign}[D_{a',a''}(\theta_h)]$. Claim 6 implies $a' \succ a''$, which proves (15).

($\Longleftarrow$) Suppose that $u : A \times \Theta \to \mathbb{R}$ has SCD, and $\succeq$ is a refinement of $\succeq_{SCD}$. For any $S \subseteq A$, define $C_u(S) \equiv \bigcup_{\theta \in \Theta} \arg\max_{a \in S} u(a, \theta)$. It is clear that

$$(\forall \theta) \quad \arg\max_{a \in S} u(a, \theta) = \arg\max_{a \in C_u(S)} u(a, \theta).$$

We claim that $C_u(S)$ is completely ordered by $\succeq_{SCD}$. To see why, take any pair $a', a'' \in C_u(S)$ with $a' \neq a''$. As u has SCD, $D_{a',a''}$ is single crossing in θ. As A is minimal, $D_{a',a''}$ is not a zero function. Also, as $a', a'' \in C_u(S)$, $\text{sign}[D_{a',a''}]$ is not a constant function with value either 1 or -1. Thus, $D_{a',a''}$ is single crossing either only from below, or only from above. It follows that $a' \succ_{SCD} a''$ or $a'' \succ_{SCD} a'$.

Since $\succeq$ is a refinement of $\succeq_{SCD}$, $\succeq$ coincides with $\succeq_{SCD}$ on $C_u(S)$, and the strong set orders generated by $\succeq$ and $\succeq_{SCD}$ on the collection of all subsets of $C_u(S)$ also coincide. By definition of $\succeq_{SCD}$, u satisfies Milgrom and Shannon's single-crossing property in (a, θ) with respect to $\succeq_{SCD}$ and $\leq$.[47] It follows from Milgrom and Shannon (1994, Theorem 4) that $\forall \theta_l < \theta_h$,

$$\arg\max_{a \in S} u(a, \theta_h) = \arg\max_{a \in C_u(S)} u(a, \theta_h) \succeq_{SSO} \arg\max_{a \in C_u(S)} u(a, \theta_l) = \arg\max_{a \in S} u(a, \theta_l).$$[48]

A.3. Proof of Theorem 5

The proof is similar to the proof of Theorem 4 in Appendix A.2.

($\Longrightarrow$) Suppose $u : A \times \Theta \to \mathbb{R}$ has MS on $(A, \succeq)$.

To show that u has SSCD on A, suppose not, per contra. As we have shown in the proof

[47] Given $(A, \succeq)$ that is completely ordered, $(\Theta, \leq)$ that is (partially) ordered, and $u : A \times \Theta \to \mathbb{R}$, Milgrom and Shannon's single-crossing property in (a, θ) is equivalent to

$$(\forall a' \succ a'')(\forall \theta_l < \theta_h) \quad u(a', \theta_l) \geq (>) u(a'', \theta_l) \implies u(a', \theta_h) \geq (>) u(a'', \theta_h).$$

[48] Milgrom and Shannon (1994, Theorem 4) identify their single-crossing property and quasi-supermodularity as jointly necessary and sufficient for their monotone comparative statics. (On a lattice $(Z, \succeq)$, $h : Z \to \mathbb{R}$ is quasisupermodular if $h(z) \geq (>) h(z \wedge z') \implies h(z \vee z') \geq (>) h(z')$.) When the choice set is completely ordered, the quasi-supermodularity holds trivially.

of Theorem 1,

$$(\exists a', a'' \text{ with } a' \neq a'')(\exists \theta_l < \theta_m < \theta_h) \quad D_{a',a''}(\theta_l) \geq 0, D_{a',a''}(\theta_m) \leq 0, \text{ and } D_{a',a''}(\theta_h) \geq 0.$$

Let $S \equiv \{a', a''\}$ and consider a selection $s^*(\theta)$ from $\arg\max_{a \in S} u(a, \theta)$ such that $s^*(\theta_l) = s^*(\theta_h) = a'$ and $s^*(\theta_m) = a''$. Since u has MS on $(A, \succeq)$, we must have $a' \succeq a''$ and $a'' \succeq a'$, a contradiction to anti-symmetry of $\succeq$.

To show that $\succeq$ is a refinement of $\succeq_{SSCD}$, it suffices to show that

$$(\forall a', a'' \in A) \quad a' \succ_{SSCD} a'' \implies a' \succ a'',$$

because both $\succeq$ and $\succeq_{SSCD}$ are anti-symmetric. Take any $a', a'' \in A$ such that $a' \succ_{SSCD} a''$. As $D_{a',a''}$ is strictly single crossing only from below, $\exists \theta_l < \theta_h$ such that $\text{sign}[D_{a',a''}(\theta_l)] < \text{sign}[D_{a',a''}(\theta_h)]$, which implies that $\text{sign}[D_{a',a''}(\theta_l)] \leq 0$ and $\text{sign}[D_{a',a''}(\theta_h)] \geq 0$. Consider a selection $a'' \in \arg\max_{a \in \{a',a''\}} u(a, \theta_l)$ and $a' \in \arg\max_{a \in \{a',a''\}} u(a, \theta_h)$. Since u has MS on $(A, \succeq)$, we have $a' \succeq a''$. Since $a' \neq a''$ and $\succeq$ is anti-symmetric, it must be that $a' \succ a''$.

($\Longleftarrow$) For any $S \subseteq A$, define $C_u(S) \equiv \bigcup_{\theta \in \Theta} \arg\max_{a \in S} u(a, \theta)$. First, we claim that $C_u(S)$ is completely ordered by $\succeq_{SSCD}$. To see this, take any pair $a', a'' \in C_u(S)$ with $a' \neq a''$. As u has SSCD , $D_{a',a''}$ is strictly single crossing in θ. As $a', a'' \in C_u(S)$, $\text{sign}[D_{a',a''}]$ is not a constant function with value either 1 or -1. Thus, $D_{a',a''}$ is strictly single crossing either only from below or only from above. It follows that $a' \succ_{SSCD} a''$ or $a'' \succ_{SSCD} a'$. Next, since $\succeq$ is a refinement of $\succeq_{SSCD}$, $\succeq$ coincides with $\succeq_{SSCD}$ on $C_u(S)$. By definition of $\succeq_{SSCD}$, u satisfies Milgrom and Shannon's strict single-crossing property in (a, θ) with respect to $\succeq_{SSCD}$ and $\leq$.[49] It follows from Milgrom and Shannon (1994, Theorem 4') that any selection $s^*(\theta)$ from $\arg\max_{a \in C_u(S)} u(a, \theta) (= \arg\max_{a \in S} u(a, \theta))$ is increasing in θ.

B. Proofs for Aggregating Single-Crossing Functions and Main Characterizations

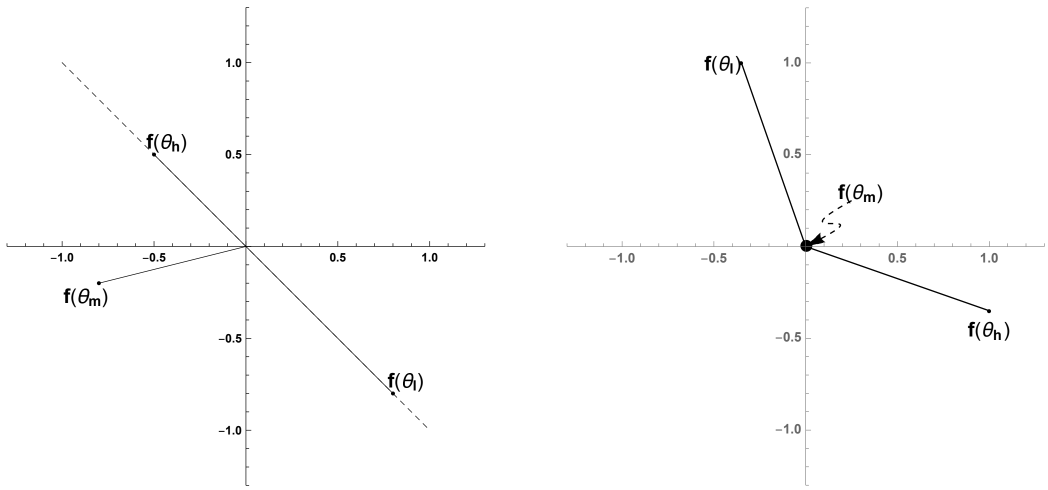
Before providing the proofs in this appendix, we first clarify Condition (5) in the definition of ratio dominance.

[49] Given $(A, \succeq)$ that is completely ordered, $(\Theta, \leq)$ that is (partially) ordered, and $u : A \times \Theta \to \mathbb{R}$, Milgrom and Shannon's strict single-crossing property in (a, θ) is equivalent to

$$(\forall a' \succ a'')(\forall \theta_l < \theta_h) \quad u(a', \theta_l) \geq u(a'', \theta_l) \implies u(a', \theta_h) > u(a'', \theta_h).$$

B.1. On the Definition of Ratio Dominance

Section 3 in the main text explained Condition (4) in the definition of ratio dominance. We impose Condition (5) to rule out cases in which, for some $\theta_l < \theta_m < \theta_h$, either (i) $f(\theta_l)$ and $f(\theta_h)$ are collinear in opposite directions while $f(\theta_m)$ is not, or (ii) $f(\theta_l)$ and $f(\theta_h)$ are non-zero vectors while $f(\theta_m)$ is not. See Figure 4, wherein panel (a) depicts case (i) and panel (b) depicts case (ii). Note that Condition (4) is satisfied in both panels.



(a) Failure of $\implies$.

(b) Failure of $\impliedby$.

Figure 4: f_1 and f_2 are not ratio ordered because Condition (5) fails for $\theta_l < \theta_m < \theta_h$.

It turns out that for all linear combinations of two single-crossing functions f_1 and f_2 to be single crossing, Condition (5) of ratio ordering is required; see the proof of Lemma 1. For example, consider a convex combination $\frac{f_1+f_2}{2}$. In panel (a), $(f_1+f_2)(\theta_l) = (f_1+f_2)(\theta_h) = 0$ while $(f_1+f_2)(\theta_m) < 0$; in panel (b), $(f_1+f_2)(\theta_l) > 0$ and $(f_1+f_2)(\theta_h) > 0$ while $(f_1+f_2)(\theta_m) = 0$.

B.2. Proof of Lemma 1

When $|\Theta| \leq 2$, the proof is trivial as all functions are single crossing and every pair of functions are ratio ordered. Hereafter, we assume $|\Theta| \geq 3$.

($\implies$) It is clear that each function f_1 and f_2 is single crossing. We must show that f_1 and f_2 are ratio ordered.

To prove (4), we suppose towards contradiction that

$$\begin{aligned}&(\exists \theta_l < \theta_h) \quad f_1(\theta_l)f_2(\theta_h) < f_1(\theta_h)f_2(\theta_l), \text{ and} \\ &(\exists \theta' < \theta'') \quad f_1(\theta')f_2(\theta'') > f_1(\theta'')f_2(\theta').\end{aligned} \tag{16}$$

Take any upper bound $\overline{\theta}$ of $\{\theta_l, \theta_h, \theta', \theta''\}$.

First, let $\alpha_l \equiv (f_2(\theta_l), -f_1(\theta_l))$. Then $(\alpha_l \cdot f)(\theta_l) = (f_2(\theta_l), -f_1(\theta_l)) \cdot (f_1(\theta_l), f_2(\theta_l)) = 0$, and $(\alpha_l \cdot f)(\theta_h) > 0$. Thus, $\alpha_l \cdot f$ is single crossing from below and $(\alpha_l \cdot f)(\overline{\theta}) > 0$.

Second, let $\alpha' \equiv (f_2(\theta'), -f_1(\theta'))$. Then $(\alpha' \cdot f)(\theta') = 0$ and $(\alpha' \cdot f)(\theta'') < 0$. Thus, $\alpha' \cdot f$ is single crossing from above and $(\alpha' \cdot f)(\overline{\theta}) < 0$.

Let $\overline{\alpha} = (f_2(\overline{\theta}), -f_1(\overline{\theta}))$. It follows that

$$\begin{aligned}&(\overline{\alpha} \cdot f)(\theta_l) = (f_2(\overline{\theta}), -f_1(\overline{\theta})) \cdot (f_1(\theta_l), f_2(\theta_l)) = -(\alpha_l \cdot f)(\overline{\theta}) < 0, \\ &(\overline{\alpha} \cdot f)(\theta') = -(\alpha' \cdot f)(\overline{\theta}) > 0, \text{ and} \\ &(\overline{\alpha} \cdot f)(\overline{\theta}) = 0.\end{aligned}$$

Therefore, $\overline{\alpha} \cdot f$ is not single crossing, a contradiction.

To prove (5), take any $\theta_l < \theta_m < \theta_h$.

First, we show that $f_1(\theta_l)f_2(\theta_h) = f_1(\theta_h)f_2(\theta_l)$ implies $f_1(\theta_m)f_2(\theta_h) = f_1(\theta_h)f_2(\theta_m)$ and $f_1(\theta_m)f_2(\theta_l) = f_1(\theta_l)f_2(\theta_m)$. Assume f_1 is not a zero function on $\{\theta_l, \theta_m, \theta_h\}$, as otherwise the proof is trivial. Since f_1 is single crossing, either $f_1(\theta_l) \neq 0$ or $f_1(\theta_h) \neq 0$. We consider the case of $f_1(\theta_h) \neq 0$ (and omit the proof for the other case, as it is analogous). Let $\alpha_h \equiv (f_2(\theta_h), -f_1(\theta_h))$. Since $\alpha_h \cdot f$ is single crossing and $(\alpha_h \cdot f)(\theta) = 0$ for $\theta = \theta_l, \theta_h$, it holds that $(\alpha_h \cdot f)(\theta_m) = f_2(\theta_h)f_1(\theta_m) - f_1(\theta_h)f_2(\theta_m) = 0$. It follows immediately that $f_1(\theta_m)f_2(\theta_h) = f_1(\theta_h)f_2(\theta_m)$. As $(f_1(\theta_m), f_2(\theta_m))$ and $(f_1(\theta_h), f_2(\theta_h))$ are linearly dependent and $(f_1(\theta_h), f_2(\theta_h))$ is a non-zero vector, there exists $\lambda \in \mathbb{R}$ such that $f_i(\theta_m) = \lambda f_i(\theta_h)$ for $i = 1, 2$. Thus,

$$f_1(\theta_l)f_2(\theta_m) = \lambda f_1(\theta_l)f_2(\theta_h) = \lambda f_2(\theta_l)f_1(\theta_h) = f_2(\theta_l)f_1(\theta_m).$$

Next, we show that if $f_1(\theta_l)f_2(\theta_m) = f_1(\theta_m)f_2(\theta_l)$ and $f_1(\theta_m)f_2(\theta_h) = f_1(\theta_h)f_2(\theta_m)$, then

$f_1(\theta_l)f_2(\theta_h) = f_1(\theta_h)f_2(\theta_l)$. Let $\alpha \equiv (f_2(\theta_l) - f_2(\theta_h), -f_1(\theta_l) + f_1(\theta_h))$. It follows that

$$(\alpha \cdot f)(\theta_l) = (f_2(\theta_l) - f_2(\theta_h))\, f_1(\theta_l) - (f_1(\theta_l) - f_1(\theta_h))\, f_2(\theta_l) = f_1(\theta_h)f_2(\theta_l) - f_1(\theta_l)f_2(\theta_h),$$
$$(\alpha \cdot f)(\theta_h) = (f_2(\theta_l) - f_2(\theta_h))\, f_1(\theta_h) - (f_1(\theta_l) - f_1(\theta_h))\, f_2(\theta_h) = f_1(\theta_h)f_2(\theta_l) - f_1(\theta_l)f_2(\theta_h), \text{ and}$$
$$(\alpha \cdot f)(\theta_m) = (f_2(\theta_l) - f_2(\theta_h))\, f_1(\theta_m) - (f_1(\theta_l) - f_1(\theta_h))\, f_2(\theta_m) = 0.$$

As $\alpha \cdot f$ is single crossing, it follows that $(\alpha \cdot f)(\theta_l) = (\alpha \cdot f)(\theta_h) = 0$, as we wanted to show.

$(\Longleftarrow)$ Assume that f_1 and f_2 are each single crossing. We prove the result for the case in which f_1 ratio dominates f_2; the other case is analogous. For any $\alpha \in \mathbb{R}^2$, we prove that $\alpha \cdot f$ is single crossing. We may assume that $\alpha \neq 0$, as the result is trivial otherwise.

Suppose, towards contradiction, that $\alpha \cdot f$ is not single crossing. Claim 5 implies there exist $\theta_l < \theta_m < \theta_h$ such that either,

$$\text{sign}[(\alpha \cdot f)(\theta_l)] < \text{sign}[(\alpha \cdot f)(\theta_m)] \text{ and } \text{sign}[(\alpha \cdot f)(\theta_m)] > \text{sign}[(\alpha \cdot f)(\theta_h)], \quad \text{or} \tag{17}$$
$$\text{sign}[(\alpha \cdot f)(\theta_l)] > \text{sign}[(\alpha \cdot f)(\theta_m)] \text{ and } \text{sign}[(\alpha \cdot f)(\theta_m)] < \text{sign}[(\alpha \cdot f)(\theta_h)]. \tag{18}$$

First, we consider the case in which $f(\theta) \equiv (f_1(\theta), f_2(\theta))$ for all $\theta \in \{\theta_l, \theta_m, \theta_h\}$ are non-zero vectors. Take any $\theta_1, \theta_2 \in \{\theta_l, \theta_m, \theta_h\}$ such that $\theta_1 < \theta_2$. As f_1 ratio dominates f_2, by Condition (4), $f(\theta_1)$ moves to $f(\theta_2)$ in a clockwise rotation with an angle less than or equal to 180 degrees. Let r_{12} be the clockwise angle from $f(\theta_1)$ to $f(\theta_2)$. The vector $\alpha \neq 0$ defines a partition of $\mathbb{R}^2$ into $\mathbb{R}^2_{\alpha,+} \equiv \{x \in \mathbb{R}^2 : \alpha \cdot x > 0\}$, $\mathbb{R}^2_{\alpha,0} \equiv \{x \in \mathbb{R}^2 : \alpha \cdot x = 0\}$, and $\mathbb{R}^2_{\alpha,-} \equiv \{x \in \mathbb{R}^2 : \alpha \cdot x < 0\}$. In both cases (17) and (18), both $f(\theta_l)$ and $f(\theta_h)$ are not in the same part of the partition that $f(\theta_m)$ belongs to. Thus, $r_{lm} > 0$ and $r_{mh} > 0$. On the other hand, both $f(\theta_l)$ and $f(\theta_h)$ are in the same closed half-space, either $\mathbb{R}^2_{\alpha,+} \cup \mathbb{R}^2_{\alpha,0}$ or $\mathbb{R}^2_{\alpha,-} \cup \mathbb{R}^2_{\alpha,0}$, and $f(\theta_m)$ is in the other closed half-space, either $\mathbb{R}^2_{\alpha,-} \cup \mathbb{R}^2_{\alpha,0}$ or $\mathbb{R}^2_{\alpha,+} \cup \mathbb{R}^2_{\alpha,0}$, respectively. Thus, $r_{lh} \geq 180$. Since Condition (4) implies $r_{lh} \leq 180$, it follows that $r_{lh} = 180$. Hence, $f(\theta_l)$ and $f(\theta_m)$ are linearly independent ($0 < r_{lm} < 180$), and similarly for $f(\theta_m)$ and $f(\theta_h)$. However, $f(\theta_l)$ and $f(\theta_h)$ are linearly dependent ($r_{lh} = 180$). This contradicts (5).

Second, suppose either $f(\theta_l) = 0$ or $f(\theta_h) = 0$. We provide the argument assuming $f(\theta_l) = 0$; it is analogous if $f(\theta_h) = 0$. Under either (17) or (18), $f(\theta_m) \neq 0$. By Condition (5), $f(\theta_m)$ and $f(\theta_h)$ are linearly dependent. In particular, because $f(\theta_m) \neq 0$, there exists a unique $\lambda \in \mathbb{R}$ such that $f(\theta_h) = \lambda f(\theta_m)$. Under either (17) or (18), $\lambda \leq 0$, which contradicts the hypothesis that f_1 and f_2 are single crossing.

Last, suppose $f(\theta_l) \neq 0$, $f(\theta_m) = 0$, and $f(\theta_h) \neq 0$. By Condition (5), $f(\theta_l)$ and $f(\theta_h)$ are linearly dependent. Hence, there exists a unique $\lambda \in \mathbb{R}$ such that $f(\theta_l) = \lambda f(\theta_h)$. Under either (17) or (18), $\lambda > 0$, which contradicts the hypothesis that f_1 and f_2 are single crossing.

B.3. Proof of Proposition 2

The result is trivial if $|Z| = 1$ and it is equivalent to Lemma 1 if $|Z| = 2$, so we may assume $|Z| \geq 3$. The proof is also straightforward if all functions $f(z, \cdot) : \Theta \to \mathbb{R}$ are multiples of one function $f(z_1, \cdot)$, i.e., if there is z_1 such that $(\exists \lambda : Z \to \mathbb{R})(\forall z) f(z, \cdot) = \lambda(z) f(z_1, \cdot)$. Thus, we further assume there exist z', z'' such that $f(z', \cdot) : \Theta \to \mathbb{R}$ and $f(z'', \cdot) : \Theta \to \mathbb{R}$ are linearly independent.

($\Longleftarrow$) Assume $f(z_1, \cdot)$ and $f(z_2, \cdot)$ are (i) each single crossing and (ii) ratio ordered, and that there are functions $\lambda_1, \lambda_2 : Z \to \mathbb{R}$ such that $(\forall z)\ f(z, \cdot) = \lambda_1(z) f(z_1, \cdot) + \lambda_2(z) f(z_2, \cdot)$. Then, for any function $\mu : Z \to \mathbb{R}$ with finite support,

$$\int_z f(z, \theta) \mathrm{d}\mu = \int_z \left(\lambda_1(z) f(z_1, \theta) + \lambda_2(z) f(z_2, \theta)\right) \mathrm{d}\mu = \sum_{i=1,2} \left(\int_z \lambda_i(z) \mathrm{d}\mu \right) f(z_i, \theta),$$

which is single crossing in θ by Lemma 1.

($\Longrightarrow$) Take any $z_1, z_2 \in Z$ such that $f_1(\cdot) \equiv f(z_1, \cdot)$ and $f_2(\cdot) \equiv f(z_2, \cdot)$ are linearly independent. Then, by Lemma 1, f_1 and f_2 are each single crossing and ratio ordered, as their linear combinations are all single crossing.

For every θ', θ'', let

$$M_{\theta', \theta''} \equiv \begin{bmatrix} f_1(\theta') & f_2(\theta') \\ f_1(\theta'') & f_2(\theta'') \end{bmatrix}.$$

We first prove the following claim:

Claim 7. *There exists $\theta_l < \theta_h$ such that* $\mathrm{rank}[M_{\theta_l, \theta_h}] = 2$.

Proof of Claim 7. As f_1 and f_2 are linearly independent, there exists θ_0 such that $f_2(\theta_0) \neq 0$. Let $\lambda \equiv -\frac{f_1(\theta_0)}{f_2(\theta_0)}$. Then, for some θ_λ, $f_1(\theta_\lambda) + \lambda f_2(\theta_\lambda) \neq 0$ and $\mathrm{rank}[M_{\theta_0, \theta_\lambda}] = 2$.

The proof is complete if $\theta_0 > \theta_\lambda$ or $\theta_0 < \theta_\lambda$. If not, take a lower and upper bound, $\underline{\theta}$ and $\overline{\theta}$, of $\{\theta_0, \theta_\lambda\}$. Then $\mathrm{rank}[M_{\underline{\theta}, \overline{\theta}}] = 2$. For otherwise, there exists $\alpha \in \mathbb{R}^2 \backslash \{0\}$ such that $M_{\underline{\theta}, \overline{\theta}} \alpha = 0$. As θ_0 and θ_λ are between $\underline{\theta}$ and $\overline{\theta}$, and $\alpha_1 f_1 + \alpha_2 f_2$ is single crossing, we have $M_{\theta_0, \theta_\lambda} \alpha = 0$, which contradicts $\mathrm{rank}[M_{\theta_0, \theta_\lambda}] = 2$. *Q.E.D.*

Now take any $z \in Z$, the function $f_z(\cdot) \equiv f(z, \cdot)$, and θ_l, θ_h in Claim 7. As $\mathrm{rank}[M_{\theta_l, \theta_h}] = 2$, the system

$$\begin{bmatrix} f_z(\theta_l) \\ f_z(\theta_h) \end{bmatrix} = \begin{bmatrix} f_1(\theta_l) & f_2(\theta_l) \\ f_1(\theta_h) & f_2(\theta_h) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \tag{19}$$

has a unique solution $\lambda \in \mathbb{R}^2$. We will show that $f_z = \lambda_1 f_1 + \lambda_2 f_2$.

Suppose, towards contradiction, there exists θ_λ such that

$$f_z(\theta_\lambda) \neq \lambda_1 f_1(\theta_\lambda) + \lambda_2 f_2(\theta_\lambda). \tag{20}$$

Let $\underline{\theta}$ and $\overline{\theta}$ respectively be a lower and an upper bound of $\{\theta_l, \theta_h, \theta_\lambda\}$. If $\text{rank}[M_{\underline{\theta},\overline{\theta}}] < 2$, there is $\lambda' \in \mathbb{R}^2\backslash\{0\}$ such that $\lambda'_1 f_1(\theta) + \lambda'_2 f_2(\theta) = 0$ for $\theta = \underline{\theta}, \overline{\theta}$. As $\lambda'_1 f_1 + \lambda'_2 f_2$ is single crossing, we have $\lambda'_1 f_1(\theta) + \lambda'_2 f_2(\theta) = 0$ for $\theta = \theta_l, \theta_h$, which contradicts $\text{rank}[M_{\theta_l,\theta_h}] = 2$.[50]

If, on the other hand, $\text{rank}[M_{\underline{\theta},\overline{\theta}}] = 2$, the system

$$\begin{bmatrix} f_z(\underline{\theta}) \\ f_z(\overline{\theta}) \end{bmatrix} = \begin{bmatrix} f_1(\underline{\theta}) & f_2(\underline{\theta}) \\ f_1(\overline{\theta}) & f_2(\overline{\theta}) \end{bmatrix} \begin{bmatrix} \lambda'_1 \\ \lambda'_2 \end{bmatrix}$$

has a unique solution $\lambda' \in \mathbb{R}^2$. As $f_z - \lambda'_1 f_1 - \lambda'_2 f_2$ is single crossing,

$$f_x(\theta_l) = \lambda'_1 f_1(\theta_l) + \lambda'_2 f_2(\theta_l) \quad \text{and} \quad f_x(\theta_h) = \lambda'_1 f_1(\theta_h) + \lambda'_2 f_2(\theta_h), \text{ and} \tag{21}$$
$$f_x(\theta_\lambda) = \lambda'_1 f_1(\theta_\lambda) + \lambda'_2 f_2(\theta_\lambda). \tag{22}$$

(21) implies that λ' solves (19). As the unique solution to (19) was λ, it follows that $\lambda' = \lambda$. But then (20) and (22) are in contradiction. Therefore, there exist $\lambda_1, \lambda_2 : Z \to \mathbb{R}$ such that

$$(\forall z, \theta) \quad f(z, \theta) = \lambda_1(z) f(z_1, \theta) + \lambda_2(z) f(z_2, \theta).$$

B.4. Proof of Theorem 2's SCD* Characterization

$(\Longleftarrow)$ Suppose that $u(a, \theta) = g_1(a) f_1(\theta) + g_2(a) f_2(\theta) + h(\theta)$, with $f_1, f_2 : \Theta \to \mathbb{R}$ each single crossing and ratio ordered. Then, for any $a, a' \in A$, $D_{a,a'}(\theta) = (g_1(a) - g_1(a')) f_1(\theta) + (g_2(a) - g_2(a')) f_2(\theta)$, which is single crossing by Lemma 1.

$(\Longrightarrow)$ Assume, without loss of generality, that $|A| \geq 2$. Take any $a_0 \in A$, and define $A' \equiv A \setminus \{a_0\}$. Define $f : A \times \Theta \to \mathbb{R}$ as $f(a, \theta) \equiv u(a, \theta) - u(a_0, \theta)$, which, for every a, is single crossing in θ.

[50] The function $\lambda'_1 f_1 + \lambda'_2 f_2$ must be single crossing because we can consider $\mu : Z \to \mathbb{R}$ such that $\mu(z_1) = \lambda'_1$, $\mu(z_2) = \lambda'_2$, and $\mu(z) = 0$ for any $z \neq z_1, z_2$. We use similar reasoning subsequently.

We will show that, for every function $\mu' : A' \to \mathbb{R}$ with finite support, $\int_{a\in A'} f(a,\theta)\mathrm{d}\mu'$ can be represented as a multiple of the difference between two convex utility combinations $\int_a u(a,\theta)\mathrm{d}P$ and $\int_a u(a,\theta)\mathrm{d}Q$. Since the environment is convex, there exist $a_P, a_Q \in A$ such that $u(a_P,\theta) = \int_a u(a,\theta)\mathrm{d}P$ and $u(a_Q,\theta) = \int_a u(a,\theta)\mathrm{d}Q$ for all θ. Since the utility difference $u(a_P,\theta) - u(a_Q,\theta)$ is single crossing, so is $\int_{a\in A'} f(a,\theta)\mathrm{d}\mu'$. The result then follows from Proposition 2.

For any function $\mu' : A' \to \mathbb{R}$ with finite support, we define a function $\mu : A \to \mathbb{R}$ as an extension of μ':

$$\mu(a_0) \equiv -\sum_{\{a:\mu'(a)\neq 0\}} \mu'(a), \text{ and } (\forall a \in A')\ \mu(a) \equiv \mu'(a).$$

In a sense, we let a_0 absorb the function values on A'. In particular, note that

$$\sum_{\{a:\mu(a)\neq 0\}} \mu(a) = \mu(a_0) + \sum_{\{a:\mu'(a)\neq 0\}} \mu(a) = 0.$$

We construct the Hahn-Jordan decomposition (μ_+, μ_-) of μ. That is, we define functions $\mu_+, \mu_- : A \to \mathbb{R}_+$ by $(\forall a \in A)\ \mu_+(a) \equiv \max\{\mu(a), 0\}$ and $\mu_-(a) \equiv -\min\{\mu(a), 0\}$. These are both functions with finite support, and $\mu = \mu_+ - \mu_-$. Let $M \equiv \sum_{\{a:\mu(a)\neq 0\}} \mu_+(a) = \sum_{\{a:\mu(a)\neq 0\}} \mu_-(a)$. If $M = 0$, pick an arbitrary $P \in \Delta A$ with finite support and let $Q = P$. If $M > 0$, define $P, Q \in \Delta A$ with probability mass functions p, q such that for any $a \in A$,

$$p(a) = \frac{\mu_+(a)}{M} \quad \text{and} \quad q(a) = \frac{\mu_-(a)}{M}.$$

Note that P and Q have finite support. Since the environment is convex, there exist $a_P, a_Q \in A$ such that $u(a_P,\theta) = \int_a u(a,\theta)\mathrm{d}P$ and $u(a_Q,\theta) = \int_a u(a,\theta)\mathrm{d}Q$ for all θ. It follows that

$$\begin{aligned}
\int_{a\in A'} f(a,\theta)\mathrm{d}\mu' &= \int_{a\in A} f(a,\theta)\mathrm{d}\mu \quad \text{(because } f(a_0,\theta) = 0\text{)} \\
&= \int_{a\in A} u(a,\theta)\mathrm{d}\mu - u(a_0,\theta)\mu(A) \\
&= \int_{a\in A} u(a,\theta)\mathrm{d}\mu_+ - \int_{a\in A} u(a,\theta)\mathrm{d}\mu_- \quad \text{(as } \mu(A) = 0\text{)} \\
&= M \cdot (u(a_P,\theta) - u(a_Q,\theta)),
\end{aligned}$$

which is single crossing.

Thus, if u has SCD*, then $f : A' \times \Theta \to \mathbb{R}$ is linear combinations SC-preserving. By Proposition 2, there exist $a_1, a_2 \in A'$ and $\lambda_1, \lambda_2 : A' \to \mathbb{R}$ such that (i) $f(a_1,\theta)$ and $f(a_2,\theta)$ are each single crossing and ratio ordered, and (ii) $(\forall a \in A')$ $f(a,\cdot) = \lambda_1(a) f(a_1,\cdot) + \lambda_2(a) f(a_2,\cdot)$. Hence, there exist functions $g_1, g_2 : A \to \mathbb{R}$ with $g_1(a_0) = g_2(a_0) = 0$ such that $(\forall a \in A)$ $f(a,\cdot) = g_1(a) f(a_1,\cdot) + g_2(a) f(a_2,\cdot)$, or equivalently,

$$(\forall a, \theta) \quad u(a,\theta) = g_1(a) f(a_1,\theta) + g_2(a) f(a_2,\theta) + u(a_0,\theta).$$

B.5. Proof of Theorem 2's SSCD* Characterization

Similar to the proof of Theorem 2 for SCD*, our proof for SSCD* requires conditions ensuring that arbitrary linear combinations of functions are strictly single crossing. We state and discuss the analogs of Lemma 1 and Proposition 2 below in Appendix B.5.1; their proofs are in Appendix B.5.2 and Appendix B.5.3 respectively. The proof of Theorem 2 for SSCD* then follows in Appendix B.5.4.

B.5.1. Aggregating Strictly Single-Crossing Functions

Lemma 2. *Let $f_1, f_2 : \Theta \to \mathbb{R}$. The linear combination $\alpha_1 f_1(\theta) + \alpha_2 f_2(\theta)$ is strictly single crossing $\forall \alpha \in \mathbb{R}^2 \backslash \{0\}$ if and only if f_1 and f_2 are strictly ratio ordered.*

Besides the change to strict single crossing and, correspondingly, strict ratio ordering, Lemma 2 has two other differences from Lemma 1. First, we rule out $(\alpha_1, \alpha_2) = 0$; this is unavoidable because a zero function is not strictly single crossing. Second, and more importantly, there is no explicit mention in Lemma 2 that f_1 and f_2 are each strictly single crossing. It turns out—as elaborated in the Lemma's proof—that when two functions are strictly ratio ordered, each of them must be strictly single crossing.

To extend Lemma 2 to more than two functions, we say that $f : Z \times \Theta \to \mathbb{R}$ is *linear combinations SSC-preserving* if $\int_z f(z,\theta) \mathrm{d}\mu$ is either a zero function or strictly single crossing in θ for every function $\mu : Z \to \mathbb{R}$ with finite support. Parallel to Proposition 2:

Proposition 3. *Let $f : Z \times \Theta \to \mathbb{R}$ for some set Z, and assume there exist $z_1, z_2 \in Z$ such that $f(z_1,\cdot) : \Theta \to \mathbb{R}$ and $f(z_2,\cdot) : \Theta \to \mathbb{R}$ are linearly independent. The function f is linear combinations SSC-preserving if and only if there exist $\lambda_1, \lambda_2 : Z \to \mathbb{R}$ such that*

1. *$f(z_1,\cdot) : \Theta \to \mathbb{R}$ and $f(z_2,\cdot) : \Theta \to \mathbb{R}$ are strictly ratio ordered, and*
2. *$(\forall z)$ $f(z,\cdot) = \lambda_1(z) f(z_1,\cdot) + \lambda_2(z) f(z_2,\cdot)$.*

For the "if" direction of Proposition 3, the existence of a pair of linearly independent functions need not be assumed, because strict ratio ordering implies linear independence. However, without that hypothesis, the "only if" direction would fail: given $Z = \{z_1, z_2\}$, and $f(z_1, \cdot) = 2f(z_2, \cdot)$ with $f(z_1, \cdot)$ strictly single crossing, the function f is linear combinations SSC-preserving even though $f(z_1, \cdot)$ and $f(z_2, \cdot)$ are not strictly ratio ordered.

B.5.2. Proof of Lemma 2

When $|\Theta| \leq 2$.

If $|\Theta| = 1$, the proof is trivial as all functions are strictly single crossing and every pair of f_1, f_2 satisfy strict ratio ordering. So assume $|\Theta| = 2$ and denote $\Theta = \{\theta_l, \theta_h\}$; without loss, we may assume $\theta_h > \theta_l$ because of our maintained assumption that upper and lower bounds exist for all pairs.

($\Longrightarrow$) Either $(f_1(\theta_l), f_2(\theta_l)) \neq 0$ or $(f_1(\theta_h), f_2(\theta_h)) \neq 0$: otherwise, for every $\alpha \in \mathbb{R}^2 \backslash \{0\}$, $(\alpha \cdot f)(\theta_l) = (\alpha \cdot f)(\theta_h) = 0$, and hence $\alpha \cdot f$ is a zero function, which is not strictly single crossing. Assume $(f_1(\theta_l), f_2(\theta_l)) \neq 0$; the proof for the other case is analogous. Let $\alpha_l \equiv (f_2(\theta_l), -f_1(\theta_l))$ and consider $(\alpha_l \cdot f)(\theta) = f_2(\theta_l)f_1(\theta) - f_1(\theta_l)f_2(\theta)$. We have $(\alpha_l \cdot f)(\theta_l) = 0$ and, by strict single crossing of $\alpha_l \cdot f$, $(\alpha_l \cdot f)(\theta_h) \neq 0$. That is, $f_2(\theta_l)f_1(\theta_h) \neq f_1(\theta_l)f_2(\theta_h)$, which means that f_1 and f_2 are strictly ratio ordered.

($\Longleftarrow$) For any $\alpha \in \mathbb{R}^2 \backslash \{0\}$, $\alpha \cdot f$ is not strictly single crossing if and only if $(\alpha \cdot f)(\theta_l) = (\alpha \cdot f)(\theta_h) = 0$. This implies $\alpha_1 f_1(\theta_l) = -\alpha_2 f_2(\theta_l)$ and $\alpha_1 f_1(\theta_h) = -\alpha_2 f_2(\theta_h)$, and hence

$$\begin{aligned}
\alpha_1 f_1(\theta_l) f_2(\theta_h) &= -\alpha_2 f_2(\theta_l) f_2(\theta_h) = \alpha_1 f_1(\theta_h) f_2(\theta_l) \quad \text{and} \\
\alpha_2 f_1(\theta_l) f_2(\theta_h) &= -\alpha_1 f_1(\theta_l) f_1(\theta_h) = \alpha_2 f_1(\theta_h) f_2(\theta_l).
\end{aligned}$$

As $(\alpha_1, \alpha_2) \neq 0$, $f_1(\theta_l)f_2(\theta_h) = f_1(\theta_h)f_2(\theta_l)$, contradicting strict ratio ordering of f_1 and f_2.

When $|\Theta| \geq 3$.

($\Longrightarrow$) Suppose, towards contradiction, that f_1 and f_2 are not strictly ratio ordered:

$$\begin{aligned}
&(\exists \theta_l < \theta_h) \quad f_1(\theta_l) f_2(\theta_h) \leq f_1(\theta_h) f_2(\theta_l) \quad \text{and} \\
&(\exists \theta' < \theta'') \quad f_1(\theta') f_2(\theta'') \geq f_1(\theta'') f_2(\theta').
\end{aligned} \tag{23}$$

Take any upper bound $\overline{\theta}$ of $\{\theta_l, \theta_h, \theta', \theta''\}$. Letting $\alpha_l \equiv (f_2(\theta_l), -f_1(\theta_l))$, it holds that $\alpha_l \cdot f$ is strictly single crossing only from below, as $(\alpha_l \cdot f)(\theta_l) = (f_2(\theta_l), -f_1(\theta_l)) \cdot (f_1(\theta_l), f_2(\theta_l)) = 0$ and by (23), $(\alpha_l \cdot f)(\theta_h) \geq 0$. Hence $(\alpha_l \cdot f)(\overline{\theta}) \geq 0$. Analogously, letting $\alpha' \equiv (f_2(\theta'), -f_1(\theta'))$, we conclude that $(\alpha' \cdot f)(\overline{\theta}) \leq 0$. Now let $\overline{\alpha} \equiv (f_2(\overline{\theta}), -f_1(\overline{\theta}))$. It follows that

$$\begin{aligned}
&(\overline{\alpha} \cdot f)(\theta_l) = (f_2(\overline{\theta}), -f_1(\overline{\theta})) \cdot (f_1(\theta_l), f_2(\theta_l)) = -(\alpha_l \cdot f)(\overline{\theta}) \leq 0, \\
&(\overline{\alpha} \cdot f)(\theta') = (f_2(\theta'), -f_1(\theta')) \cdot (f_1(\theta'), f_2(\theta')) = -(\alpha' \cdot f)(\overline{\theta}) \geq 0, \text{ and} \\
&(\overline{\alpha} \cdot f)(\overline{\theta}) = 0.
\end{aligned}$$

Therefore, $\overline{\alpha} \cdot f$ is not strictly single crossing.

($\Longleftarrow$) We provide a proof for the case in which f_1 strictly ratio dominates f_2, and omit the other case's analogous proof. For any $\alpha \in \mathbb{R}^2 \backslash \{0\}$, we prove that $\alpha \cdot f$ is single crossing. The argument is very similar to that used in proving Lemma 1, but note that here we do not assume that f_1 and f_2 are each strictly single crossing.

As f_1 strictly ratio dominates f_2,

$$(\forall \theta_l < \theta_h) \quad f_1(\theta_l) f_2(\theta_h) < f_1(\theta_h) f_2(\theta_l). \tag{24}$$

Suppose, towards contradiction, that $\alpha \cdot f$ is not strictly single crossing.

Claim: There exist $\theta_l, \theta_m, \theta_h$ with $\theta_l < \theta_m < \theta_h$ such that

$$(\alpha \cdot f)(\theta_l) \leq 0, (\alpha \cdot f)(\theta_m) \geq 0, \text{ and } (\alpha \cdot f)(\theta_h) \leq 0, \quad \text{or} \tag{25}$$

$$(\alpha \cdot f)(\theta_l) \geq 0, (\alpha \cdot f)(\theta_m) \leq 0, \text{ and } (\alpha \cdot f)(\theta_h) \geq 0. \tag{26}$$

Proof of claim: Since $\alpha \cdot f$ is not strictly single crossing either from below or from above,

$$\begin{aligned}
&(\exists \theta_1 < \theta_2) \quad (\alpha \cdot f)(\theta_1) \geq 0 \geq (\alpha \cdot f)(\theta_2), \text{ and} \\
&(\exists \theta_3 < \theta_4) \quad (\alpha \cdot f)(\theta_3) \leq 0 \leq (\alpha \cdot f)(\theta_4).
\end{aligned}$$

Let $\Theta_0 \equiv \{\theta_1, \theta_2, \theta_3, \theta_4\}$ and let $\overline{\theta}$ and $\underline{\theta}$ be an upper and lower bound of Θ_0, respectively. Either $(\alpha \cdot f)(\underline{\theta}) \neq 0$ or $(\alpha \cdot f)(\overline{\theta}) \neq 0$, as otherwise $f_1(\underline{\theta}) f_2(\overline{\theta}) = f_2(\underline{\theta}) f_1(\overline{\theta})$, contradicting (24). Suppose $(\alpha \cdot f)(\overline{\theta}) \neq 0$. If $(\alpha \cdot f)(\overline{\theta}) < 0$, then we choose $(\theta_l, \theta_m, \theta_h) = (\theta_3, \theta_4, \overline{\theta})$, which satisfies (25). If $(\alpha \cdot f)(\overline{\theta}) > 0$, then we choose $(\theta_l, \theta_m, \theta_h) = (\theta_1, \theta_2, \overline{\theta})$, which satisfies (26). A similar argument applies when $(\alpha \cdot f)(\underline{\theta}) \neq 0$. $\|$

Condition (24) implies that $f(\theta) \equiv (f_1(\theta), f_2(\theta)) \neq 0$ for all $\theta \in \{\theta_l, \theta_m, \theta_h\}$. Take any $\theta_1, \theta_2 \in \{\theta_l, \theta_m, \theta_h\}$ such that $\theta_1 < \theta_2$. By (24), $f(\theta_1)$ moves to $f(\theta_2)$ in a clockwise rotation with an angle $r_{12} \in (0, 180)$. Suppose (25) holds; the argument is analogous if (26) holds. It follows from $0 < r_{lh} < 180$, $(\alpha \cdot f)(\theta_l) \leq 0$, and $(\alpha \cdot f)(\theta_h) \leq 0$ that $\{f(\theta_l), f(\theta_h)\} \subseteq \mathbb{R}^2_{\alpha,-} \cup \mathbb{R}^2_{\alpha,0}$

with $\{f(\theta_l), f(\theta_h)\} \not\subseteq \mathbb{R}^2_{\alpha,0}$.[51] This, together with $r_{lm} > 0$ and $r_{mh} > 0$, implies $f(\theta_m) \in \mathbb{R}^2_{\alpha,-}$, which contradicts (25).

B.5.3. Proof of Proposition 3

Appendix B.3 proved Proposition 2 assuming certain functions are linearly independent. Essentially the same proof can be used for Proposition 3, replacing statements involving "single crossing" with "either a zero function or strictly single crossing".

B.5.4. Proof of the SSCD* Portion of Theorem 2

The utility function $u : A \times \Theta \to \mathbb{R}$ has SSCD* if and only if $(\forall a, a' \in A)$ $D_{a,a'}$ is either a zero function or strictly single crossing. Most statements in the proof of Theorem 2 for SCD* go through for SSCD* when we replace "single crossing" with "either a zero function or strictly single crossing".

We need only to rewrite the proof of the "only if" part in the following two special cases:

1. $(\forall a', a'')(\forall \theta)$ $u(a', \theta) = u(a'', \theta)$, or
2. $(\exists a', a'')$ such that (i) $u(a'', \theta) - u(a', \theta)$ is not a zero function of θ, and (ii) $(\forall a)$ $u(a, \theta) - u(a', \theta)$ and $u(a'', \theta) - u(a', \theta)$ are linearly dependent functions of θ.

In the first case, we can write $u(a, \theta)$ in form of (6) where g_1, g_2 are zero functions, $h(\theta) \equiv u(a_0, \theta)$ for any a_0, $(\forall \theta)$ $f_1(\theta) = 1$, and $f_2(\theta)$ is any strictly decreasing function of θ. Then,

$$(\forall \theta_l < \theta_h) \quad f_1(\theta_l) f_2(\theta_h) = f_2(\theta_h) < f_2(\theta_l) = f_1(\theta_h) f_2(\theta_l).$$

In the second case, for every a, there exists $\lambda \in \mathbb{R}^2 \backslash \{0\}$ such that $\lambda_1 \left(u(a, \cdot) - u(a', \cdot)\right) + \lambda_2 \left(u(a'', \cdot) - u(a', \cdot)\right)$ is a zero function. Note that $\lambda_1 \neq 0$, as otherwise $u(a'', \cdot) - u(a', \cdot)$ would be a zero function. It follows that there exists $\lambda : A \to \mathbb{R}$ such that

$$(\forall a, \theta) \quad u(a, \theta) - u(a', \theta) = \lambda(a) \left(u(a'', \theta) - u(a', \theta)\right),$$

or equivalently,

$$(\forall a, \theta) \quad u(a, \theta) = \lambda(a) \left(u(a'', \theta) - u(a', \theta)\right) + u(a', \theta).$$

Note that $u(a'', \theta) - u(a', \theta)$ is a strictly single-crossing function of θ: consider the expectational difference with distributions that put probability one on a'' and a' respectively. If the difference is strictly single crossing from below, we can write $u(a, \theta)$ in the form of (6) where

[51] Recall that $\mathbb{R}^2_{\alpha,+} \equiv \{x \in \mathbb{R}^2 : \alpha \cdot x > 0\}$, $\mathbb{R}^2_{\alpha,0} \equiv \{x \in \mathbb{R}^2 : \alpha \cdot x = 0\}$, and $\mathbb{R}^2_{\alpha,-} \equiv \{x \in \mathbb{R}^2 : \alpha \cdot x < 0\}$.

$g_1(a) = \lambda(a)$, $g_2(a) = 0$, $f_1(\theta) = u(a'', \theta) - u(a', \theta)$, and $h(\theta) = u(a', \theta)$. If the difference is strictly single crossing only from above, we let $g_1(a) = -\lambda(a)$ and $f_1(\theta) = u(a', \theta) - u(a'', \theta)$. Now take any strictly increasing function $h : \Theta \to \mathbb{R}$ and define

$$\hat{h}(\theta) \equiv \begin{cases} -e^{h(\theta)} & \text{if } f_1(\theta) \le 0 \\ e^{-h(\theta)} & \text{otherwise} \end{cases} \quad \text{and} \quad f_2(\theta) \equiv \begin{cases} \hat{h}(\theta) f_1(\theta) & \text{if } f_1(\theta) \neq 0 \\ 1 & \text{otherwise.} \end{cases}$$

To verify that f_1 and f_2 are strictly ratio ordered, take any $\theta_l < \theta_h$. There are three possibilities to consider:

1. If $f_1(\theta_l) f_1(\theta_h) > 0$, then

$$f_1(\theta_l) f_2(\theta_h) = f_1(\theta_l) f_1(\theta_h) \hat{h}(\theta_h) < f_1(\theta_l) f_1(\theta_h) \hat{h}(\theta_l) = f_1(\theta_h) f_2(\theta_l),$$

   as $\hat{h}(\theta)$ is strictly decreasing over $\{\theta \mid f_1(\theta) < 0\}$ and $\{\theta \mid f_1(\theta) > 0\}$.

2. If $f_1(\theta_l) f_1(\theta_h) < 0$, then as $f_1(\theta)$ is strictly single crossing from below, we have $f_1(\theta_l) < 0 < f_1(\theta_h)$. Hence,

$$f_1(\theta_l) f_2(\theta_h) = f_1(\theta_l) f_1(\theta_h) \hat{h}(\theta_h) < 0 < f_1(\theta_l) f_1(\theta_h) \hat{h}(\theta_l) = f_1(\theta_h) f_2(\theta_l).$$

3. If $f_1(\theta_l) f_1(\theta_h) = 0$, then because f_1 is strictly single crossing from below, we have either (i) $f_1(\theta_l) < 0 = f_1(\theta_h)$, which results in $f_1(\theta_l) f_2(\theta_h) = f_1(\theta_l) < 0 = f_1(\theta_h) f_2(\theta_l)$, or (ii) $f_1(\theta_l) = 0 < f_1(\theta_h)$, which results in $f_1(\theta_l) f_2(\theta_h) = 0 < f_1(\theta_h) = f_1(\theta_h) f_2(\theta_l)$.

B.6. Proof of Proposition 1

We prove the first part of Proposition 1 for SCD* and omit an analogous proof of the second part for SSCD*. If $|\Theta| \le 2$, then the proof is trivial, so assume $|\Theta| \ge 3$. The "if" direction of the result follows directly from Theorem 2: if u has a positive affine transformation $\tilde{u}$ of the form in Proposition 1, then u, as a positive affine transformation of $\tilde{u}$, has SCD*.

For the "only if" direction, take any u that has SCD*. Following the form given in Theorem 2, a positive affine transformation of u is

$$\tilde{u}(a, \theta) = g_1(a) f_1(\theta) + g_2(a) f_2(\theta),$$

where $f_1, f_2 : \Theta \to \mathbb{R}$ are each single crossing and ratio ordered.

First, we consider the case in which $f(\underline{\theta})$ and $f(\overline{\theta})$ are linearly dependent.[52] Assume, with a positive affine transformation of $\tilde{u}$, that the length of the vector $f(\theta) \equiv (f_1(\theta), f_2(\theta))$ in $\mathbb{R}^2$ is either 0 or 1 for every θ. If $f(\underline{\theta}) = f(\overline{\theta}) = 0$, then because f_1 and f_2 are each single crossing, we have $(\forall\theta)\ f_1(\theta) = f_2(\theta) = 0$ and $(\forall a, \theta)\ \tilde{u}(a, \theta) = 0$. We can easily now rewrite $\tilde{u}$ in the form (7), with $\lambda : \Theta \to [0, 1]$ increasing.

Suppose $f(\overline{\theta}) \neq 0$; we omit the analogous proof for the case of $f(\underline{\theta}) \neq 0$. By Condition (5) of ratio ordering, for every θ, the vector $f(\theta) \in \mathbb{R}^2$ is linearly dependent on $f(\overline{\theta})$. As $(\forall\theta)\ \|f(\theta)\| \in \{0, 1\}$, there exists $\lambda : \Theta \to \{-1, 0, 1\}$ such that $(\forall\theta)\ f(\theta) = \lambda(\theta) f(\overline{\theta})$. Note that λ is increasing because f_1 and f_2 are each single crossing. If either $\lambda(\underline{\theta}) = 0$ (and so $(\forall a)\ \tilde{u}(a, \underline{\theta}) = 0$) or $\lambda(\underline{\theta}) = 1$ (and so $(\forall\theta)\ \lambda(\theta) = 1$), then

$$\tilde{u}(a, \theta) = \lambda(\theta)\tilde{u}(a, \overline{\theta}) + (1 - \lambda(\theta))\tilde{u}(a, \underline{\theta}),$$

with the last term equal to zero. If, on the other hand, $\lambda(\underline{\theta}) = -1$, then

$$\begin{aligned}
\tilde{u}(a, \theta) = \lambda(\theta)\tilde{u}(a, \overline{\theta}) &= \frac{\lambda(\theta) + 1}{2}\tilde{u}(a, \overline{\theta}) + \frac{\lambda(\theta) - 1}{2}\left(-\tilde{u}(a, \underline{\theta})\right) \\
&= \frac{\lambda(\theta) + 1}{2}\tilde{u}(a, \overline{\theta}) + \left(1 - \frac{\lambda(\theta) + 1}{2}\right)\tilde{u}(a, \underline{\theta}).
\end{aligned}$$

Next, suppose that the vectors $f(\underline{\theta}), f(\overline{\theta}) \in \mathbb{R}^2$ are linearly independent, so the angle between the vectors is strictly less than 180 degrees. As f_1 and f_2 are ratio ordered, for each θ there exists $\alpha(\theta), \beta(\theta) \in \mathbb{R}_+$ such that

$$f(\theta) = \alpha(\theta) f(\overline{\theta}) + \beta(\theta) f(\underline{\theta}),$$

or equivalently,

$$\tilde{u}(a, \theta) = \alpha(\theta)\tilde{u}(a, \overline{\theta}) + \beta(\theta)\tilde{u}(a, \underline{\theta}).$$

By Condition (5), $f(\theta) \neq 0$, which implies that $\alpha(\theta) + \beta(\theta) > 0$. A positive affine transformation of dividing $\hat{v}(\cdot, \theta)$ by $\alpha(\theta) + \beta(\theta)$ results in the form (7), where $\lambda(\theta) \equiv \frac{\alpha(\theta)}{\alpha(\theta) + \beta(\theta)} \in [0, 1]$.

To prove that the function λ is increasing, take θ_1, θ_2 such that $\underline{\theta} \leq \theta_1 \leq \theta_2 \leq \overline{\theta}$. To reduce notation below, let $\alpha_i \equiv \alpha(\theta_i)$ and $\beta_i \equiv \beta(\theta_i)$ for $i = 1, 2$. We must show that $\frac{\alpha_1}{\alpha_1 + \beta_1} \leq \frac{\alpha_2}{\alpha_2 + \beta_2}$, or equivalently that $\alpha_1 \beta_2 \leq \alpha_2 \beta_1$. Suppose f_1 ratio dominates f_2; the other

[52] This case can be ignored in the proof for SSCD*, because if f_1 and f_2 are strictly ratio ordered, then $f(\underline{\theta})$ and $f(\overline{\theta})$ must be linearly independent.

case is analogous. Then $f_1(\theta_1)f_2(\theta_2) \leq f_1(\theta_2)f_2(\theta_1)$, and hence

$$\left(\alpha_1 f_1(\overline{\theta}) + \beta_1 f_1(\underline{\theta})\right)\left(\alpha_2 f_2(\overline{\theta}) + \beta_2 f_2(\underline{\theta})\right) \leq \left(\alpha_2 f_1(\overline{\theta}) + \beta_2 f_1(\underline{\theta})\right)\left(\alpha_1 f_2(\overline{\theta}) + \beta_1 f_2(\underline{\theta})\right),$$

or equivalently,

$$(\alpha_1\beta_2 - \alpha_2\beta_1)\left(f_1(\overline{\theta})f_2(\underline{\theta}) - f_1(\underline{\theta})f_2(\overline{\theta})\right) \leq 0.$$

Note that $f_1(\overline{\theta})f_2(\underline{\theta}) - f_1(\underline{\theta})f_2(\overline{\theta}) > 0$ because f_1 ratio dominates f_2, and $f(\underline{\theta})$ and $f(\overline{\theta})$ are linearly independent. Hence, $\alpha_1\beta_2 \leq \alpha_2\beta_1$.

C. Proofs for the Implications of Theorem 2

C.1. Proof of Corollary 2

It is clear from Corollary 1 that $v(x,\theta) = -|x-\theta|^2 = -x^2 + 2x\theta - \theta^2$ has SCED, as $f_1(\theta) = -1$ and $f_2(\theta) = 2\theta$ are each single crossing and ratio ordered, and we take $g_1(x) = x^2$, $g_2(x) = x$, and $h(\theta) = -\theta^2$.

For the converse, it is sufficient to prove the following claim.

Claim 8. *If there exist $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ and $f_1, f_2, h : \Theta \to \mathbb{R}$ such that*

$$v(x,\theta) \equiv -|x-\theta|^z = g_1(x)f_1(\theta) + g_2(x)f_2(\theta) + h(\theta),$$

then $z = 2$.

Proof of Claim 8. Fix $x_0 \in \mathbb{R}$ and define $\tilde{v}(x,\theta) \equiv v(x,\theta) - v(x_0,\theta) = \tilde{g}_1(x)f_1(\theta) + \tilde{g}_2 f_2(\theta)$, where $\tilde{g}_1(x) \equiv g_1(x) - g_1(x_0)$ and $\tilde{g}_2 \equiv g_2(x) - g_2(x_0)$. Fix any $\theta_l < \theta_m < \theta_h$. There exists $(\lambda_l, \lambda_m, \lambda_h) \in \mathbb{R}^3\backslash\{0\}$ such that

$$\begin{bmatrix} f_1(\theta_l) & f_1(\theta_m) & f_1(\theta_h) \\ f_2(\theta_l) & f_2(\theta_m) & f_2(\theta_h) \end{bmatrix} \begin{bmatrix} \lambda_l \\ \lambda_m \\ \lambda_h \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Hence, for every $x \in \mathbb{R}$,

$$\begin{aligned} h(x) &\equiv \lambda_l \tilde{v}(x,\theta_l) + \lambda_m \tilde{v}(x,\theta_m) + \lambda_h \tilde{v}(x,\theta_h) \\ &= \begin{bmatrix} \tilde{g}_1(x) & \tilde{g}_2(x) \end{bmatrix} \begin{bmatrix} f_1(\theta_l) & f_1(\theta_m) & f_1(\theta_h) \\ f_2(\theta_l) & f_2(\theta_m) & f_2(\theta_h) \end{bmatrix} \begin{bmatrix} \lambda_l \\ \lambda_m \\ \lambda_h \end{bmatrix} = 0. \end{aligned}$$

We hereafter consider $\lambda_l \neq 0$ (and omit the proofs for the other two cases, $\lambda_m \neq 0$ and $\lambda_h \neq 0$, which are analogous). The previous equation implies that for any $x \in \mathbb{R}$,

$$\tilde{v}(x, \theta_l) = -\frac{\lambda_m}{\lambda_l}\tilde{v}(x, \theta_m) - \frac{\lambda_h}{\lambda_l}\tilde{v}(x, \theta_h). \tag{27}$$

At any $x < \theta$, $\tilde{v}(x, \theta) = -(\theta - x)^z - v(x_0, \theta)$ is differentiable in x, and hence (27) implies that the partial derivative $\tilde{v}_x(x, \theta_l)$ exists at $x = \theta_l$. Thus, the right partial derivative $\lim_{\varepsilon \downarrow 0} \frac{\tilde{v}(\theta_l + \varepsilon, \theta_l) - \tilde{v}(\theta_l, \theta_l)}{\varepsilon} = -\lim_{\varepsilon \downarrow 0} \varepsilon^{z-1}$ must equal the left partial derivative $\lim_{\varepsilon \downarrow 0} \frac{\tilde{u}(\theta_l - \varepsilon, \theta_l) - \tilde{v}(\theta_l, \theta_l)}{-\varepsilon} = \lim_{\varepsilon \downarrow 0} \varepsilon^{z-1}$, which implies $\lim_{\varepsilon \downarrow 0} \varepsilon^{z-1} = 0$, and thus $z > 1$.

Now suppose to contradiction that $z \neq 2$. At any $x > \theta_h$, (27) and $\tilde{v}(x, \theta) = -(x - \theta)^z - v(x_0, \theta)$ imply

$$-\lambda_l (x - \theta_l)^z = \lambda_m (x - \theta_m)^z + \lambda_h (x - \theta_h)^z + (\lambda_m + \lambda_h - \lambda_l) v(x_0, \theta),$$

and hence, differentiating with respect to x and simplifying using $z > 1$ and $z \neq 2$:

$$-\lambda_l (x - \theta_l)^{z-1} = \lambda_m (x - \theta_m)^{z-1} + \lambda_h (x - \theta_h)^{z-1}, \tag{28}$$

$$-\lambda_l (x - \theta_l)^{z-2} = \lambda_m (a - \theta_m)^{z-2} + \lambda_h (x - \theta_h)^{z-2}, \tag{29}$$

$$-\lambda_l (x - \theta_l)^{z-3} = \lambda_m (x - \theta_m)^{z-3} + \lambda_h (x - \theta_h)^{z-3}. \tag{30}$$

It follows that $\lambda_m \lambda_h \neq 0$: if, for example, $\lambda_m = 0$, then (28) implies $\lambda_h \neq 0$ (as $\lambda_l \neq 0$), and then (28) and (29) imply $x - \theta_l = x - \theta_h$ for all $x > \theta_h$, contradicting $\theta_l < \theta_h$. Since $\left((x - \theta_l)^{z-2}\right)^2 = (x - \theta_l)^{z-1}(x - \theta_l)^{z-3}$, we manipulate the right-hand sides of (28)–(30) to obtain

$$2\lambda_m \lambda_h (x - \theta_m)^{z-2} (x - \theta_h)^{z-2} = \lambda_m \lambda_h \left((x - \theta_m)^{z-1} (x - \theta_h)^{z-3} + (x - \theta_m)^{z-3} (x - \theta_h)^{z-1} \right),$$

which simplifies, using $\lambda_m \lambda_h \neq 0$, to

$$2 = \frac{x - \theta_h}{x - \theta_m} + \frac{x - \theta_m}{x - \theta_h}.$$

Therefore, $x - \theta_h = x - \theta_m$ for all $x > \theta_h$, contradicting $\theta_m < \theta_h$. *Q.E.D.*

C.2. Proof of Corollary 5

Proof. The "if" direction of the result follows directly from Theorem 2. For the "only if" direction, we apply Theorem 2 and observe that, for any $a', a'' \in A$,

$$D_{a',a''}(\theta) = (g^I(a') - g^I(a''))f^I(\theta) + (g^{II}(a') - g^{II}(a''))f^{II}(\theta),$$

with some $f^I, f^{II} : \Theta \to \mathbb{R}$ each single-crossing and ratio ordered, and $g^I, g^{II} : A \to \mathbb{R}$.

For each dimension i for which $g_i : A_i \to \mathbb{R}$ is non-constant, we take $a', a'' \in A$ with $g_i(a'_i) \neq g_i(a''_i)$ and $a'_j = a''_j$ for $j \neq i$. It follows that $D_{a',a''}(\theta) = (g_i(a'_i) - g_i(a''_i))f_i(\theta)$, and letting $\lambda_i^I \equiv \frac{g^I(a') - g^I(a'')}{g_i(a'_i) - g_i(a''_i)}$ and $\lambda_i^{II} \equiv \frac{g^{II}(a') - g^{II}(a'')}{g_i(a'_i) - g_i(a''_i)}$, that

$$(\forall \theta) \quad f_i(\theta) = \lambda_i^I f^I(\theta) + \lambda_i^{II} f^{II}(\theta).$$

For each dimension i for which $g_i : A_i \to \mathbb{R}$ is constant, we set $\lambda_i^I \equiv 0$ and $\lambda_i^{II} \equiv 0$.

We have

$$u(a, \theta) = \sum_{i=1}^n g_i(a) f_i(\theta) = \left(\sum_{i=1}^n \lambda_i^I g_i(a_i) \right) f^I(\theta) + \left(\sum_{i=1}^n \lambda_i^{II} g_i(a_i) \right) f^{II}(\theta) + h(\theta),$$

where $h(\theta) \equiv \sum_{i \in \{j : g_j \text{ is constant}\}} g_i f_i(\theta)$, with each g_i a constant in the summation. *Q.E.D.*

C.3. Proof of Corollary 6

$(\Longleftarrow)$ For any $Q \in A \equiv \left\{ P \in \Delta\Delta\Omega : \int_{p \in \Delta\Omega} p \mathrm{d}P = p^* \right\}$,

$$u(Q, \theta) = \left(\int_{\Delta\Omega} g_1(p) \mathrm{d}Q \right) f_1(\theta) + \left(\int_{\Delta\Omega} g_2(p) \mathrm{d}Q \right) f_2(\theta) + \int_{\Delta\Omega} \left(\sum_{\omega \in \Omega} v(\delta_\omega, \theta) p(\omega) \right) \mathrm{d}Q.$$

The last term on the right-hand side is equal to

$$\sum_\omega v(\delta_\omega, \theta) \left(\int_{\Delta\Omega} p(w) \mathrm{d}Q \right) = \sum_\omega v(\delta_\omega, \theta) p^*(\omega).$$

Thus, for any $Q, R \in A$,

$$D_{Q,R}(\theta) = \left(\int_{\Delta\Omega} g_1(p) \mathrm{d}Q - \int_{\Delta\Omega} g_1(p) \mathrm{d}R \right) f_1(\theta) + \left(\int_{\Delta\Omega} g_2(p) \mathrm{d}Q - \int_{\Delta\Omega} g_2(p) \mathrm{d}R \right) f_2(\theta),$$

which is single crossing in θ by Lemma 1.

($\Longrightarrow$) Suppose that v has SCED-X with a full-support prior p^*. By Theorem 2, for any experiment $Q \in A$,

$$u(Q,\theta) = \int_{p\in\Delta\Omega} v(p,\theta)\mathrm{d}Q = g_1(Q)f_1(\theta) + g_2(Q)f_2(\theta) + h(\theta),$$

where $g_1, g_2 : A \to \mathbb{R}$, $h : \Theta \to \mathbb{R}$, and $f_1, f_2 : \Theta \to \mathbb{R}$ are each single crossing and ratio ordered.

Take any posterior $p \in \Delta\Omega$, and find $\alpha \in (0,1]$ and $q \in \Delta\Omega$ such that $p^* = \alpha p + (1-\alpha)q$. We consider two experiments: Q_p yields posteriors p and q with probability α and $1-\alpha$, respectively, and R_p yields each degenerate posterior $\delta_\omega \in \Delta\Omega$ with probability $\alpha p(\omega)$, and posterior q with probability $1-\alpha$. Observe that $Q_p, R_p \in A$. Thus,

$$u(Q_p,\theta) = \alpha v(p,\theta) + (1-\alpha)v(q,\theta) = g_1(Q_p)f_1(\theta) + g_2(Q_p)f_2(\theta) + h(\theta), \quad \text{and}$$
$$u(R_p,\theta) = \alpha\left(\sum_\omega v(\delta_\omega,\theta)p(\omega)\right) + (1-\alpha)v(q,\theta) = g_1(R_p)f_1(\theta) + g_2(R_p)f_2(\theta) + h(\theta).$$

Hence, $u(Q_p,\theta) - u(R_p,\theta) = \alpha\left(v(p,\theta) - \sum_\omega v(\delta_\omega,\theta)p(\omega)\right)$, which implies that

$$v(p,\theta) - \sum_\omega v(\delta_\omega,\theta)p(\omega) = \tilde{g}_1(p)f_1(\theta) + \tilde{g}_2(p)f_2(\theta),$$

where $\tilde{g}_i(p) = \frac{g_i(Q_p)-g_i(R_p)}{\alpha}$ for $i = 1,2$.

D. Proofs for MD*

Appendix D.1 states analogs of Lemma 1 and Proposition 2 for monotonic functions; their proofs are in Appendix D.2 and Appendix D.3 respectively. The proof of Theorem 3 then follows in Appendix D.4.

D.1. Aggregating Monotonic Functions

Lemma 3. *Let $f_1, f_2 : \Theta \to \mathbb{R}$ be monotonic functions. The linear combination $\alpha_1 f_1(\theta) + \alpha_2 f_2(\theta)$ is monotonic $\forall \alpha \in \mathbb{R}^2$ if and only if either f_1 or f_2 is an affine transformation of the other, i.e., there exists $\lambda \in \mathbb{R}^2$ such that either $f_2 = \lambda_1 f_1 + \lambda_2$ or $f_1 = \lambda_1 f_2 + \lambda_2$.*

We say that $f : Z \times \Theta \to \mathbb{R}$ is *linear combinations monotonicity-preserving* if $\int_z f(z,\theta)\mathrm{d}\mu$ is a monotonic function of θ for every function $\mu : Z \to \mathbb{R}$ with finite support.

Proposition 4. *Let $f : Z \times \Theta \to \mathbb{R}$ for some set Z. The function f is linear combinations monotonicity-preserving if and only if there exist $z' \in Z$ and $\lambda_1, \lambda_2 : Z \to \mathbb{R}$ such that (i) $f(z', \cdot)$ is monotonic, and (ii) $(\forall z)\ f(z, \cdot) = \lambda_1(z) f(z', \cdot) + \lambda_2(z)$.*

D.2. Proof of Lemma 3

$(\Longleftarrow)$ Suppose there exist $\lambda \in \mathbb{R}^2$ such that $f_2 = \lambda_1 f_1 + \lambda_2$. Then, for any $\alpha \in \mathbb{R}^2$,

$$(\alpha \cdot f)(\theta) = \alpha_1 f_1(\theta) + \alpha_2(\lambda_1 f_1(\theta) + \lambda_2) = (\alpha_1 + \alpha_2 \lambda_1) f_1(\theta) + \lambda_2,$$

which is monotonic.

$(\Longrightarrow)$ The proof is trivial if both f_1 and f_2 are constant functions. So we suppose that at least one function, say f_1, is not constant:

$$(\exists \theta', \theta'') \quad f_1(\theta') \neq f_1(\theta''). \tag{31}$$

This implies that $\operatorname{rank}[M_{\theta',\theta''}] = 2$, where

$$M_{\theta',\theta''} \equiv \begin{bmatrix} f_1(\theta') & 1 \\ f_1(\theta'') & 1 \end{bmatrix}.$$

Hence, the system

$$\begin{bmatrix} f_2(\theta') \\ f_2(\theta'') \end{bmatrix} = \begin{bmatrix} f_1(\theta') & 1 \\ f_1(\theta'') & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \tag{32}$$

has a unique solution $\lambda^* \in \mathbb{R}^2$. We will show that $f_2 = \lambda_1^* f_1 + \lambda_2^*$.

Suppose, towards contradiction, there exists θ^* such that

$$f_2(\theta^*) \neq \lambda_1^* f_1(\theta^*) + \lambda_2^*. \tag{33}$$

Let $\underline{\theta}$ and $\overline{\theta}$ be a lower and upper bound of $\{\theta', \theta'', \theta^*\}$. If $\operatorname{rank}[M_{\underline{\theta},\overline{\theta}}] < 2$, then $f_1(\overline{\theta}) = f_1(\underline{\theta})$. As θ' and θ'' are in between $\underline{\theta}$ and $\overline{\theta}$ and f_1 is monotonic, we have $f_1(\theta') = f_1(\theta'')$, which contradicts (31). If, on the other hand, $\operatorname{rank}[M_{\underline{\theta},\overline{\theta}}] = 2$, then the system

$$\begin{bmatrix} f_2(\underline{\theta}) \\ f_2(\overline{\theta}) \end{bmatrix} = \begin{bmatrix} f_1(\underline{\theta}) & 1 \\ f_1(\overline{\theta}) & 1 \end{bmatrix} \begin{bmatrix} \lambda_1' \\ \lambda_2' \end{bmatrix}$$

has a unique solution $\lambda' \in \mathbb{R}^2$. As θ', θ'', and θ^* are in between $\underline{\theta}$ and $\overline{\theta}$, and $f_2 - \lambda_1' f_1$ is monotonic, we have

$$\begin{bmatrix} f_2(\theta') \\ f_2(\theta'') \end{bmatrix} = \begin{bmatrix} f_1(\theta') & 1 \\ f_1(\theta'') & 1 \end{bmatrix} \begin{bmatrix} \lambda_1' \\ \lambda_2' \end{bmatrix} \text{ and} \tag{34}$$

$$f_2(\theta^*) = \lambda_1' f_1(\theta^*) + \lambda_2'. \tag{35}$$

Equation 34 implies that λ' solves (32). As the unique solution to (32) was λ^*, it follows that $\lambda' = \lambda^*$. But then (33) and (35) are in contradiction.

D.3. Proof of Proposition 4

($\Longleftarrow$) We omit the proof as it is similar to the proof of Proposition 2 in Appendix B.3.

($\Longrightarrow$) For the proof of necessity, if $(\forall z)$ $f(z,\theta)$ is a constant function of θ, then we let $\lambda_1(z) = 0$ and $\lambda_2(z) = f(z,\theta)$. If there exists $z' \in Z$ such that $f(z',\theta)$ is not a constant function of θ, then Lemma 3 implies $(\forall z,\theta)$ $f(z,\theta) = \lambda_1(z) f(z',\theta) + \lambda_2(z)$, with $\lambda_1, \lambda_2 : Z \to \mathbb{R}$.

D.4. Proof of Theorem 3

($\Longleftarrow$) We omit the proof as it is similar to the proof of Theorem 2 in Appendix B.4.

($\Longrightarrow$) The proof is trivial if $(\forall a,\theta)$ $u(a,\theta) = 0$, so assume there exists a_0 such that $u(a_0,\cdot) : \Theta \to \mathbb{R}$ is not a zero function. Define $f : A \times \Theta \to \mathbb{R}$ by $f(a,\theta) \equiv u(a,\theta) - u(a_0,\theta)$. Note that $(\forall a)$ $f(a,\theta)$ is a monotonic function of θ.

Let $A' \equiv A \backslash \{a_0\}$. As in the proof of Theorem 2 in Appendix B.4, for every $\mu' : A' \to \mathbb{R}$ with finite support, there exist convex utility combinations $\int_a u(a,\theta) \mathrm{d}P$ and $\int_a u(a,\theta) \mathrm{d}Q$, where P and Q have finite support, such that $\int_{a \in A'} f(a,\theta) \mathrm{d}\mu'$ is monotonic if and only if $\int_{a \in A} u(a,\theta) \mathrm{d}P - \int_{a \in A} u(a,\theta) \mathrm{d}Q$ is monotonic. Since the environment is convex, the latter utility difference is indeed monotonic, and so $\int_{a \in A'} f(a,\theta) \mathrm{d}\mu'$ is monotonic. By Proposition 4, there exist $a' \in A \backslash a_0$ and $\lambda_1, \lambda_2 : A \backslash \{a_0\} \to \mathbb{R}$ such that $(\forall a,\theta)$ $f(a,\theta) = \lambda_1(a) f(a',\theta) + \lambda_2(a)$. Hence, there exist functions $g_1, g_2 : A \to \mathbb{R}$ with $g_1(a_0) = g_2(a_0) = 0$ such that $f(a,\theta) = g_1(a) f(a',\theta) + g_2(a)$, or equivalently, $u(a,\theta) = g_1(a) f(a',\theta) + g_2(a) + u(a_0,\theta)$.

E. Relationship to Signed-Ratio Monotonicity

Quah and Strulovici (2012) establish that for any two functions $f_1 : \Theta \to \mathbb{R}$ and $f_2 : \Theta \to \mathbb{R}$ that are each single crossing from below, $\alpha_1 f_1 + \alpha_2 f_2$ is single crossing from below for all $\alpha \in \mathbb{R}^2_+$ if and only if f_1 and f_2 satisfy *signed-ratio monotonicity*: for all $i,j \in \{1,2\}$,

$$(\forall \theta_l < \theta_h) \quad f_j(\theta_l) < 0 < f_i(\theta_l) \implies f_i(\theta_h) f_j(\theta_l) \leq f_i(\theta_l) f_j(\theta_h). \tag{36}$$

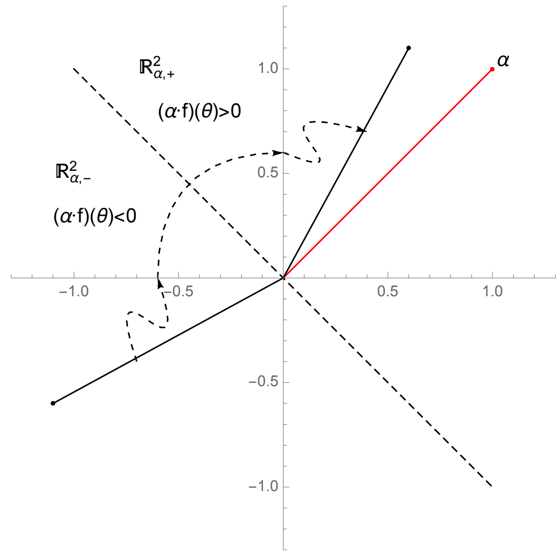


Figure 5: Signed-ratio monotonicity and single crossing of a convex combination.

Given our discussion in Subsection 3.1 of a graphical interpretation of ratio ordering, one can see that Condition (36) implies that the vector $f(\theta) \equiv (f_1(\theta), f_2(\theta))$ rotates clockwise as θ increases within the upper-left quadrant (i.e., when $f_1(\cdot) < 0 < f_2(\cdot)$), while it rotates counterclockwise within the lower-right quadrant (i.e., when $f_1(\cdot) > 0 > f_2(\cdot)$); there are no restrictions in the other two quadrants.[53] The dashed curve with arrowheads in Figure 5 provides a depiction. Note that if f_1 and f_2 are both single crossing from below (or both from above), then there cannot exist $\theta_l < \theta_h$ such that one of $f(\theta_l)$ and $f(\theta_h)$ is in the upper-left quadrant and the other in the lower-right quadrant. It follows that if f_1 and f_2 are both single crossing from below, then ratio ordering implies signed-ratio monotonicity; more generally, however, the implication is not valid.

Figure 5 also illustrates Quah and Strulovici's (2012) result, analogous to Figure 2 for Lemma 1. Any linear combination $\alpha \in \mathbb{R}^2_+ \backslash \{0\}$ defines two open half spaces, $\mathbb{R}^2_{\alpha,-} \equiv \{x \in \mathbb{R}^2 : \alpha \cdot x < 0\}$ and $\mathbb{R}^2_{\alpha,+} \equiv \{x \in \mathbb{R}^2 : \alpha \cdot x > 0\}$, as indicated in Figure 5. If the vector $f(\theta)$ rotates monotonically as θ increases from $\mathbb{R}^2_{\alpha,-}$ to $\mathbb{R}^2_{\alpha,+}$, or either half space contains the vector $f(\theta)$ for all θ, then $\alpha \cdot f \equiv \alpha_1 f_1 + \alpha_2 f_2$ is single crossing from below. Conversely, if $f(\theta)$ does not rotate monotonically in the upper-left or lower-right quadrant, then there exists $\alpha \in \mathbb{R}^2_+ \backslash \{0\}$ such that $\alpha \cdot f$ is not single crossing from below.

[53] To be precise: by "quadrant" we mean the interiors, i.e., excluding the axes.

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