Pricing for Coordination

Marina Halac   Elliot Lipnowski   Daniel Rappoport*

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Abstract

A seller prices a good with network externalities. Purchasing decisions being complementary, a pricing policy can yield equilibrium multiplicity. We study how personalized pricing can be used to mitigate this strategic uncertainty, guaranteeing a high revenue. An optimal policy offers personalized discounts to successively insulate against low-demand equilibria, and posts a high price to extract revenue from the induced higher demand. The result is price dispersion and a higher quantity of trade than would occur if the seller could choose her preferred equilibrium. We examine how the optimal policy changes with the strength of externalities and heterogeneity across buyer groups.

*Halac: Department of Economics, Yale University, 30 Hillhouse Avenue, New Haven, CT 06511, and CEPR (email: marina.halac@yale.edu); Lipnowski: Department of Economics, Yale University, 30 Hillhouse Avenue, New Haven, CT 06511 (email: elliot.lipnowski@yale.edu); Rappoport: University of Chicago Booth School of Business, 5807 South Woodlawn Avenue, Chicago, IL 60637 (email: daniel.rappoport@chicagobooth.edu).
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1. Introduction

We study a seller who sells a good to a population of buyers, with two key features. First, there is incomplete information: a buyer's value from purchasing the good depends on some privately known characteristic, namely his type. Second, there are network externalities: a buyer's value from purchasing the good increases with the number of other buyers who also purchase it.

There are many applications for this canonical setting. Incomplete information is the quintessential feature of the monopoly problem, and network externalities are prevalent across industries. Take, for example, a multiplayer game or social media platform. The utility a consumer derives from joining the platform depends on his privately known propensity to play online games or seek social interactions, and also increases with the number of users he can reach via the platform. Similar considerations apply to sellers of file-sharing services, dating websites, and payment apps, among others. In finance, these features are central to a firm raising capital. An investor's incentive to invest with the firm depends on his other planned investments, which are his private information, and is higher if more other investors invest, as the firm is then more likely to be successful.

The seller's problem is to offer each buyer a price to maximize revenue. Prices can be personalized—e.g., via discounts and promotional deals directed to different buyers—but they cannot be conditioned on buyers' types, which are hidden. Buyers decide whether to purchase given the seller's price offers and their types, and given their expectations of other buyers' purchasing decisions. Due to the externalities in consumption, a pricing policy can yield multiple outcomes, with a high total quantity of trade if buyers anticipate that many others will purchase, or a low total quantity if buyers are less optimistic about others' purchases. Low-quantity outcomes are naturally bad for revenue.

Our main result characterizes the optimal pricing policy that guarantees the seller a high revenue, i.e. that maximizes revenue in her worst-case outcome. This policy takes the form of a posted price with dispersed discounts. The seller offers personalized discounts to (some) buyers to successively insulate against low-demand outcomes, and posts a high price to extract revenue from the induced higher demand. List prices together with targeted discounts that vary across buyers are common in applications. Even if this personalization is partly based on customer data, the allocation of discounts is often arbitrary to a significant extent.[1] Our analysis provides a rationale for these practices and sheds light on their comparative statics.

To illustrate the seller's problem and our main results, we next describe an example that is a special case of our model. Suppose there is a unit mass of ex-ante identical buyers with types drawn uniformly from $\Theta = [0, 1]$. The seller offers a price $p_i \in \mathbb{R}_+$ to each buyer i, and then buyers simultaneously decide whether to purchase. If a buyer of type $\theta_i \in \Theta$ purchases at a price p_i and the total quantity demanded (i.e., the total mass of buyers who purchase) is $q \in [0, 1]$, then the buyer's payoff is $\theta_i q - p_i$. The buyer purchases if, given the total quantity he anticipates, this payoff is weakly greater than his payoff from not purchasing, which is 0. Summarizing the seller's price offers by their distribution $\Pi \in \Delta(\mathbb{R}_+)$, a total quantity q is an equilibrium quantity given Π if it is the quantity demanded when buyers anticipate it.[2]

Suppose first that the seller sets only one price. That is, Π is degenerate, taking the form of a posted price. We show in Proposition 1 that this policy is optimal under best-case selection, namely if the seller were able to pick the equilibrium that buyers play whenever multiple equilibria arise. In this case, the seller would post a price $p^{\mathrm{B}} \approx 0.22$, yielding a best-case equilibrium with total quantity $q^{\mathrm{B}} \approx 0.66$ and revenue $R^{\mathrm{B}} \approx 0.148$. However, given this or any

[1] For instance, in multiplayer gacha games, the use of loot boxes and the so-called "pity system" essentially yield a maximum list price together with random discounts that accrue to only some of the players (see, e.g., Gan, 2023). As another example, the dating app Tinder was found to charge users different prices with much of the differences being unaccounted for by clear explanatory variables (see, e.g., European Commission, 2024). More broadly, numerous merchants use Groupon to provide discounts and build their customer base; selective discounts help them attract consumers via both word-of-mouth marketing and network externalities (see `https://www.groupon.com/merchant/working-with-groupon/merchant-success-stories`).

[2] Given Π, if buyers anticipate a total quantity q, then the quantity demanded is $D_q(\Pi) := \int (1 - p/q)\, \mathrm{d}\Pi(p)$. The quantity q is an equilibrium quantity if $D_q(\Pi) = q$.

other strictly positive posted price, there is also an equilibrium with zero total quantity—no buyer is willing to purchase at such a price if they anticipate that no other buyer will purchase. This policy thus performs poorly under worst-case selection: the seller's guaranteed revenue is zero.

It is clear that the seller must offer some buyers a price of 0 to ensure a positive demand. How about then just setting two prices? The seller can offer a price 0 to a share $\pi \in (0, 1)$ of the population and some price p^* to the remaining $1 - \pi$ share. Buyers anticipate that at least π buyers will purchase, so setting $p^* \in (0, \pi)$ guarantees a positive revenue. In fact, we can verify that an optimal two-price distribution has $\pi \approx p^* \approx 0.25$, yielding a worst-case equilibrium quantity $q \approx 0.75$ and revenue $R \approx 0.125$. But why two prices and not more? For example, the seller could choose a uniform price distribution, $\Pi(p) = p/p^*$ for $p \in [0, p^*]$ and some $p^* > 0$ (perturbed to add some small mass at price 0). The optimal such distribution has $p^* \approx 0.41$, yielding a worst-case equilibrium quantity $q \approx 0.71$ and revenue $R \approx 0.126$.

Our results show that the seller's optimal price distribution in this example is indeed uniform, but only up to a mass point at the top. This distribution is given by $\Pi^*(p) = 2p$ for $p \in [0, p^*)$ and $\Pi^*(p) = 1$ for $p > p^*$, where $p^* \approx 0.28$. We provide an illustration in Figure 1. The worst-case equilibrium under Π^* has total quantity $q^* \approx 0.72$ and revenue $R^* \approx 0.133$.

The shape of the optimal price distribution reflects two goals of the seller. On the one hand, the seller wishes to ensure a high demand. We show that the optimal way to do so is by using a greedy function that builds the demand from the bottom, placing as little mass on low prices as is needed to iteratively rule out low-demand outcomes. This greedy function is uniform in the example above. On the other hand, the seller also wishes to extract revenue given the induced demand. The optimal way to do so is with a posted price, hence the mass point at the highest supported price p^*.

Theorem 1 provides a characterization for our general model. The main primitive of our model is the distribution over buyers' willingness to pay given an anticipated total quantity. We identify concavity conditions on this primitive under which the seller's optimal price distribution is greedy up to its

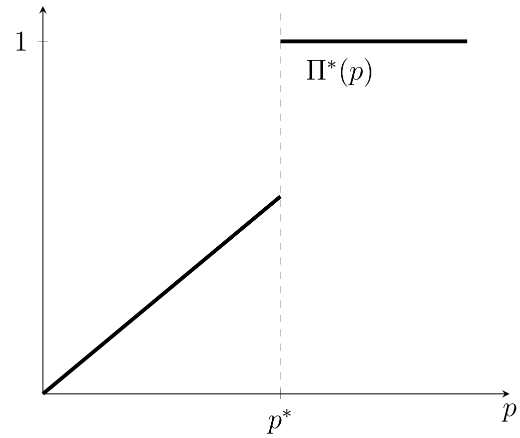

Figure 1: Optimal price distribution in the example described in the Introduction.

highest supported price, with a mass point at that price. A key point in our proof is that contractions of the price distribution that preserve demand given an anticipated quantity increase both demand and revenue given any higher anticipated quantity. This is why a posted price obtains under best-case selection, and why greediness pins down price dispersion under worst-case selection. We prove that the greedy function—which corresponds to the solution to an integral equation—is continuous and strictly increasing. Thus, the seller's optimal policy can be interpreted as a posted price with (fully) dispersed discounts.

We use our characterization of the optimal policy to study the effects of externalities. The seller's concern for strategic uncertainty in the presence of externalities results in price dispersion, with some buyers receiving generous discounts. We show however that the maximum offered price is higher than the best-case posted price, and thus not all buyers benefit from the seller's worst-case focus. The example described above, where $p^* > p^{\mathrm{B}}$, provides an illustration. At the same time, we establish that the seller's solution induces a higher total quantity of trade than the best-case benchmark—observe $q^* > q^{\mathrm{B}}$ in the example. This ranking holds even though the seller extracts lower revenue from any given total quantity under worst-case selection.[3] An implication

[3] Our finding that $q^* > q^{\mathrm{B}}$ contrasts with results in complete-information settings; see

is that buyers on average do purchase at lower prices in the worst-case solution; indeed, we show that consumer surplus is larger than in the best-case benchmark.

Turning to comparative statics, we define externalities to be stronger if buyers' values grow with the anticipated quantity demanded at a higher rate, and are thus higher for any given anticipated quantity. We find that the stronger the externalities, the less weight the seller's optimal price distribution puts on low prices, and the higher the total quantity that she induces. Our characterization also applies to a population where buyers belong to observable groups of heterogeneous strength of externalities. Similar to applications where data-based personalization is possible, here discounts can be allocated to buyers based on their group, as well as arbitrarily within each group. We show that the seller offers larger discounts to weak-externality buyers in order to build demand and extract higher revenue from strong-externality buyers.

We conclude with a discussion of variants of our model and potential avenues for future research. In this paper, we focus on a simple model that introduces externalities into an otherwise standard monopoly setting. We believe this framework can be enriched in a number of directions—for example, incorporating congestion, dynamics, and two-sided markets—to shed further light on the use of pricing for coordination.

Literature. Our paper relates to three main literatures. First, there is a sizable literature on monopoly pricing under incomplete information but in the absence of externalities. Most useful to our analysis is Bulow and Roberts (1989), which relates concepts from optimal auction design (as in Myerson, 1981) to the problem of third-degree price discrimination under capacity constraints. We build on their insights to solve our benchmark problem of best-case selection in Section 3.

Second, there is also a large literature on markets with network externalities. Classic references include Rohlfs (1974), which highlights the possibility of multiple equilibria under a posted price, and Katz and Shapiro (1985, 1986)

Segal (2003).

and Ellison and Fudenberg (2000), which study models of technology adoption with potentially incompatible products/upgrades. Oren, Smith and Wilson (1982), Csorba (2008), Aoyagi (2013), and Veiga (2018) consider settings more similar to ours, but the former two focus on second-degree price discrimination, and the latter two allow for multilateral schemes that condition on the number of buyers.[4] We are not aware of work in this literature that studies optimal personalized pricing (or more general bilateral contracts; see Section 6) under incomplete information—neither with best-case nor with worst-case selection.

Third, our paper belongs to a growing literature on contracting with externalities that focuses on worst-case selection (or unique implementation). Following respectively the seminal contributions of Segal (2003) and Winter (2004), one strand of this literature studies settings where agents' actions are bilaterally contractible, as in our model, while another strand examines moral hazard problems with unobservable actions. Within the first strand, Halac, Kremer and Winter (2020) consider agents with heterogeneous but observable attributes.[5,6] Our main departure from this literature is that we study a monopoly setting in which agents' attributes are hidden.[7] We discuss how our analysis would change under complete information in Section 6.

Finally, in addition to these literatures, we relate to papers that predict pricing policies similar to the ones we characterize but in quite different environments. For example, Perry (1984) studies an incumbent firm that seeks

[4] Such multilateral schemes are also the focus of Dybvig and Spatt (1983).

[5] Bernstein and Winter (2012) and Sákovics and Steiner (2012) examine related models with observable heterogeneity. More broadly related, Chan (2024) studies different implementation requirements in a general setting where agents' payoffs constitute a weighted potential game; Ali, Haghpanah, Lin and Siegel (2022) and Gan and Li (2024) consider worst-case selection for a seller of information that faces equilibrium multiplicity due to market expectations; and a number of papers study coordination via exclusionary contracts, including Rasmusen, Ramseyer and Wiley (1991), Innes and Sexton (1994), Segal and Whinston (2000), Spiegler (2000), and Genicot and Ray (2006).

[6] Within the second strand, see, e.g., Eliaz and Spiegler (2015); Moriya and Yamashita (2020); Halac, Lipnowski and Rappoport (2021, 2022); Cusumano, Gan and Pieroth (2023); Camboni and Porcellacchia (2024); Halac, Kremer and Winter (forthcoming).

[7] Che and Spier (2008) consider a two-agent example with hidden information in their analysis of coordination in settlement offers. In the aforementioned paper by Ali et al. (2022), the seller sets a non-personalized disclosure fee under endogenous incomplete information about buyer values.

to prevent entry and can post different prices for different units of its total supply. The firm uses a continuum of prices, with unlimited supply at the top and just enough supply at each lower price to make entry unattractive. In Heidhues and Kőszegi (2014), a monopolist sells to a loss-averse consumer who forms expectations prior to purchasing based on the monopolist's announced price distribution. To lure the consumer and exploit his attachment, an optimal distribution combines a continuum of sale prices with an atom at a high price. Our paper provides a complementary theory that emphasizes the role of externalities in consumption. These externalities and the strategic uncertainty they generate determine the optimal form of price dispersion in our model.

2. Model

Our model introduces strategic complementarities into a canonical monopoly setting. Below we describe the setup, the seller's problem, and our assumptions. We also provide examples of special cases.

2.1. Setup

We study a seller who sells a good to a population of buyers, each with a unit demand. Buyers' identities $i \in I := [0,1]$ are uniformly distributed and independent of their payoff types $\theta \in \Theta$. The seller makes a price offer $p_i \in \mathbb{R}_+$ to each buyer $(i,\theta) \in I \times \Theta$. Prices are personalized, namely they can depend on a buyer's identity i. The offered price however cannot condition on a buyer's type θ, which is the buyer's private information.[8]

Given the price offers, the buyers simultaneously decide whether or not to purchase from the seller. Denote the total quantity of purchases—i.e., the total mass of buyers who purchase—by $q \in [0,1]$. If a buyer of type $\theta \in \Theta$ purchases at a price p_i and the total purchased quantity is q, the buyer gets a payoff of

$$u(\theta, q) - p_i. \tag{1}$$

[8] Since a buyer's type is independent of his identity, it is also independent of his price offer. Using this fact, we show in Section 6 that our focus on personalized price offers is without loss of generality within the class of public bilateral contracts.

The measurable function $u : \Theta \times [0,1] \to \mathbb{R}_+$ is increasing in its second argument, reflecting that buyers' purchasing decisions are complementary. A buyer who does not purchase gets a payoff of 0.

The random variable $u(\cdot, q)$ represents a buyer's willingness to pay given an anticipated total quantity q. Let $F_q : \mathbb{R} \to [0,1]$ denote its cumulative distribution function. We assume F_q has support $[0, \overline{v}(q)] \subset \mathbb{R}_+$ for $q \in [0,1]$, where $\overline{v}$ is continuously differentiable with $\overline{v}' > 0$. This says that the lowest value is zero, whereas the highest value is strictly increasing in anticipated quantity. We further make the "cold-start" assumption that $\overline{v}(0) = 0$, so a buyer's value is almost surely zero if he anticipates no other buyer will purchase.[9] For strictly positive anticipated quantity $q \in (0,1]$, we suppose F_q admits a density f_q which is strictly positive on $(0, \overline{v}(q)]$, and that $f_q(v)$ is continuous in (q, v) where $0 \leq v \leq \overline{v}(q)$, having a partial derivative with respect to q that is also continuous in (q, v) over this domain.

Given a fixed anticipated quantity $q \in [0,1]$, the quantity that buyers demand and the seller's revenue can be easily computed. Assume that a buyer who is indifferent over purchasing chooses to purchase.[10] Then, given anticipated quantity q, the **quantity demanded** from a price p is equal to the mass of buyers whose willingness to pay is weakly greater than p, denoted $D_q(p) := 1 - F_q(p^-)$, and the quantity demanded from a price distribution $\Pi \in \Delta(\mathbb{R}_+)$ is $D_q(\Pi) := \int D_q(p)\, \mathrm{d}\Pi(p)$. Similarly, the seller's **revenue** from a price p is $R_q(p) := pD_q(p)$, and her revenue from a price distribution Π is $R_q(\Pi) := \int R_q(p)\, \mathrm{d}\Pi(p)$.[11]

2.2. Seller's problem

The seller's price offers $(p_i)_{i \in I}$ induce a coordination game between the buyers. In this game, each buyer (i, θ) simultaneously makes a decision of whether to purchase, with his payoff from purchasing given by (1). Since a

[9] We relax the zero-lowest-value and cold-start assumptions in Section 6.

[10] Without this tie-breaking assumption, our results would remain unchanged if we allow the seller to use (slightly) negative prices for a small fraction of the buyer population.

[11] In a slight abuse of notation, we let $D_q(\Pi)$ and $R_q(\Pi)$ be similarly defined by such integrals for any function $\Pi : \mathbb{R}_+ \to \mathbb{R}$ such that the integral is well defined.

buyer's identity conveys no information about his type, we can summarize the seller's price offers by their distribution $\Pi \in \Delta(\mathbb{R}_+)$. Given such a price distribution Π, if all buyers anticipate a total quantity q, the total quantity demanded is $D_q(\Pi)$. Thus, a total quantity q is an **equilibrium quantity** given Π if it is the quantity demanded when buyers anticipate it: $D_q(\Pi) = q$.

Due to the complementarity in buyers' purchasing decisions, multiple equilibrium quantities may arise given a price distribution Π. The seller wishes to guarantee a high revenue, and is therefore concerned with maximizing revenue in the worst-case equilibrium. Formally, her optimal value is given by

$$\sup_{\Pi \in \Delta(\mathbb{R}_+)} \min_{q \in [0,1]} R_q(\Pi) \quad \text{subject to} \quad D_q(\Pi) = q. \tag{P}$$

Lemma 1 and Lemma 2 in the Appendix verify that the objective is continuous and the set of equilibrium quantities is closed and nonempty. We say that (Π^*, q^*) is **optimal** if there exists a sequence $(\Pi_k, q_k)_k$ that converges to (Π^*, q^*) such that quantity q_k is the worst-case equilibrium quantity given price distribution Π_k for every k and $R_{q_k}(\Pi_k)$ converges to the seller's optimal value in (P).

Remark 1. The complementarity in buyers' purchasing decisions implies that the seller's revenue R_q is increasing in q. Hence, a worst-case equilibrium and a best-case equilibrium for the seller are respectively a lowest-quantity equilibrium and a highest-quantity equilibrium, and these equilibria exist for a given price distribution (see Lemma 2 in the Appendix).

Remark 2. While we have stated the seller's problem as maximizing revenue in the worst-case equilibrium, in our setting this is equivalent to maximizing revenue in the worst-case rationalizable outcome. The reason is that buyers' purchasing decisions are complementary, and thus the game they play under any price distribution is supermodular. The equivalence then follows from Guesnerie and Jara-Moroni (2011), who extend results of Milgrom and Roberts (1990) to games with a continuum of players. Further building on this observation and given Assumption 2 below, it will also turn out that the seller's worst-case problem is essentially equivalent to a more constrained one that maximizes revenue subject to inducing a unique equilibrium.

Remark 3. We have set up the model as one with positive externalities (i.e., with buyers' payoffs increasing in q). This implies that the worst-case equilibrium for the seller is also the worst-case equilibrium for the buyers. However, virtually nothing in our analysis changes if we assume that a buyer's payoff from purchasing is 0 while that from not purchasing is $-u(\theta, q)$.[12] In this case, there are negative externalities on nontraders (cf. Segal, 1999), and the seller's worst-case equilibrium is the best-case equilibrium for the buyers.

2.3. Concavity assumptions

We make three assumptions that we maintain throughout our analysis. Observe that while it is natural to describe our model in terms of the buyers' willingness-to-pay function $u(\theta, q)$ and the distribution of buyers' types $\theta \in \Theta$ (as we will do when providing examples in Section 2.4), there is a sense in which this is over-specified. In fact, different pairs of function $u(\theta, q)$ and distribution of θ map to the same distribution F_q over buyers' willingness to pay and therefore yield the same equilibrium conditions. Below, we thus express our model assumptions in terms of our model primitive F_q.

Our first two assumptions concern the shape of externalities in buyers' purchasing decisions. Our model is one in which externalities are positive (but see Remark 3) and increasing (i.e., purchasing decisions are strategic complements).[13] Our first assumption strengthens the sense in which externalities are increasing by requiring a monotone likelihood ratio property (MLRP):

Assumption 1 (MLRP). *For any* $0 < q \leq \tilde{q} \leq 1$*, the likelihood ratio* $f_{\tilde{q}}(v)/f_q(v)$ *is weakly increasing in* v *over* $(0, \overline{v}(q)]$.

Our model assumption that $u(\cdot, q)$ is increasing in q says that the distribution of willingness to pay under an anticipated quantity $\tilde{q}$ first-order stochastically dominates that under any lower anticipated quantity $q \leq \tilde{q}$, and MLRP

[12] The only result that changes in this case is the consumer surplus claim in Proposition 3.

[13] This terminology follows Segal (2003).

requires such dominance even when conditioning on any set of values. Using the Arrow-Pratt equivalence, this implies that for any price distribution $\Pi \in \Delta(\mathbb{R}_+)$ and quantities $0 < q \leq \tilde{q} \leq 1$, the demand $D_{\tilde{q}}(\Pi)$ is a concave transformation of the demand $D_q(\Pi)$ over the common support $[0, \overline{v}(q)]$.

Our second assumption requires the demand function to be strictly concave in anticipated quantity.

Assumption 2 (Concave externalities). *Wherever $q \in [0,1]$ and $p \in \mathbb{R}_{++}$ have $p < \overline{v}(q)$, the demand function $D_q(p)$ is strictly concave in q.*

This assumption can be interpreted as saying that externalities in our model are concave: a buyer's payoff from purchasing relative to not purchasing increases with the anticipated total quantity demanded at a decreasing rate.

Finally, for our third assumption, we define the **cross virtual value** associated with a buyer's willingness to pay v. For any $0 < q \leq \tilde{q} \leq 1$, the cross virtual value function $\varphi_{q,\tilde{q}} : (0, \overline{v}(q)] \to \mathbb{R}$ is given by

$$\varphi_{q,\tilde{q}}(v) := \frac{f_{\tilde{q}}(v)}{f_q(v)} \left[v - \frac{1 - F_{\tilde{q}}(v)}{f_{\tilde{q}}(v)} \right].$$

This function is exactly the Myerson virtual value function under total quantity $\tilde{q}$ in the special case that $q = \tilde{q}$, and is otherwise the Myerson virtual value function under $\tilde{q}$ normalized by the likelihood ratio $f_{\tilde{q}}(v)/f_q(v)$. Recall that Myerson regularity says that the virtual value function $\varphi_{\tilde{q},\tilde{q}}$ is increasing. We make an analogous assumption on the cross virtual value function:

Assumption 3 (Cross regularity). *For any $0 < q \leq \tilde{q} \leq 1$, the cross virtual value function $\varphi_{q,\tilde{q}}(v)$ is strictly increasing in v over $(0, \overline{v}(q)]$.*

For intuition, fix an actual total quantity $\tilde{q}$. As noted by Bulow and Roberts (1989), the Myerson virtual value function corresponds to the seller's marginal revenue, and thus Myerson regularity implies that the seller's revenue is concave in the quantity demanded. Cross regularity serves an analogous role, but applies across different hypothetical anticipated quantities $q \leq \tilde{q}$. In particular, it implies that for any price distribution $\Pi \in \Delta(\mathbb{R}_+)$ and anticipated and actual total quantities $0 < q \leq \tilde{q} \leq 1$, the revenue $R_{\tilde{q}}(\Pi)$ is a concave transformation of the demand $D_q(\Pi)$ over the common support $[0, \overline{v}(q)]$.

2.4. Examples

We will illustrate our results with the following special cases of our model.

Linear demand. Suppose a buyer's willingness to pay given type $\theta \in \Theta$ and anticipated quantity $q \in [0,1]$ is $u(\theta, q) = \theta\overline{v}(q)$ for $\overline{v}$ satisfying $1/\overline{v}(q)$ strictly convex in q over $(0,1]$ (as well as $\overline{v}(0) = 0$ and $\overline{v}' > 0$), and let buyers' types be drawn uniformly from $\Theta = [0,1]$. The condition on $\overline{v}$ is equivalent to our concave externalities assumption; it holds, for example, if $\overline{v}$ is log-concave. The demand function takes the linear form $D_q(p) = 1 - p/\overline{v}(q)$, and one can verify that all of our model assumptions are satisfied. Our comparative-static results in Section 5 will focus on this environment.

Proportional values. Suppose a buyer's willingness to pay given type $\theta \in \Theta$ and anticipated quantity $q \in [0,1]$ is $u(\theta, q) = \theta q$. Denoting by G the distribution of buyers' types, with positive density g, we can then rewrite our assumptions as follows. MLRP and cross regularity say, respectively, that for all $\alpha > 1$, the ratio $g(\theta)/g(\alpha\theta)$ is weakly increasing in θ and the cross virtual value function

$$\frac{g(\theta)}{g(\alpha\theta)}\left[\theta - \frac{1 - G(\theta)}{g(\theta)}\right]$$

is strictly increasing in θ wherever $g(\alpha\theta)$ is strictly positive. Concave externalities says that $G(p/q)$ is convex in q. An example that satisfies these conditions is the power distribution with $g(\theta) = \kappa\theta^{\kappa-1}$ for $\kappa \geq 1$ and $\Theta = [0,1]$.

Other examples. The examples described above take a willingness-to-pay function of the form $u(\theta, q) = \theta\overline{v}(q)$ for a buyer type $\theta \in \Theta$ and anticipated quantity $q \in [0,1]$. Our model can also accommodate other formulations; for example, $u(\theta, q) = e^{\theta q} - 1$ paired with a uniform distribution of types over $\Theta = [0,1]$ would satisfy all of our assumptions.

A natural setting that is outside our model as stated is one in which a buyer's willingness to pay takes an additive form, $u(\theta, q) = \theta + q$. This formulation does not satisfy our zero-lowest-value and cold-start assumptions. Both of these assumptions however can be relaxed—see Section 6 for details—and our main takeaways remain valid in settings like the additive one.

3. Benchmark: best-case selection

Before we solve the seller's problem in (P), we consider a benchmark setting in which the seller has no concern for strategic uncertainty. Suppose that for any price distribution $\Pi \in \Delta(\mathbb{R}_+)$ that the seller chooses, she can select the equilibrium that buyers play in the induced game if multiple equilibria arise. Rather than being concerned with the worst case as in (P), such a seller maximizes revenue in the best-case equilibrium:

$$\sup_{\Pi \in \Delta(\mathbb{R}_+)} \max_{q \in [0,1]} R_q(\Pi)$$
$$\text{subject to} \quad D_q(\Pi) = q.$$

We find that the seller's solution under best-case selection takes the form of a posted price.

Proposition 1. *Under best-case selection, some optimum exists, and any optimum has strictly positive revenue and degenerate price distribution.*

The argument builds on Bulow and Roberts (1989). Recall that by cross regularity (Assumption 3)—in fact, by Myerson regularity, which is implied by cross-regularity—the seller's revenue is concave in the quantity demanded. This means that revenue increases if price dispersion is reduced in a way that keeps quantity unchanged. Hence, given any nondegenerate price distribution Π and best-case equilibrium quantity $q > 0$ that it induces, the seller can improve upon Π with a q-preserving posted price, namely by offering each buyer a price $p = D_q^{-1}(q) \in [0, \overline{v}(q)]$.

The posted-price mechanism is a familiar result in the monopoly setting, as it is the one that obtains in the absence of externalities. If, for a fixed quantity $q \in (0, 1]$, our seller faced an exogenous distribution F_q of buyer values and thus an exogenous demand curve D_q, she would maximize revenue by offering the same price, call it $p^M(q)$, to each buyer. No equilibrium multiplicity would arise in such a setting with no externalities, and by (cross) regularity, the optimal posted price would be the unique $p^M \in (0, \overline{v}(q))$ with $\varphi_{q,q}(p^M) = 0$.

Proposition 1 tells us that the presence of externalities does not alter the nature of the seller's optimal mechanism provided that the seller can select her preferred equilibrium. The seller uses a posted price as in a standard monopoly setting, although naturally the externalities do affect the price she chooses. The complementarity in buyers' purchasing decisions implies that the total quantity demanded is more responsive to price changes. Thus, letting p^B and q^B be respectively an optimal price and equilibrium quantity that solve the best-case problem above, one can show that $p^B \leq p^M(q^B)$, with strict inequality under sufficient smoothness conditions.[14]

4. Optimal price distribution

We now return to the problem in (P), where the seller is concerned with worst-case outcomes. The seller chooses a price distribution to maximize revenue in her least preferred equilibrium of the induced game between the buyers. In Section 4.1, we present an auxiliary program that clarifies the key constraints that the worst-case focus introduces. We then use this auxiliary program in Section 4.2 to derive our main result on the seller's optimal price distribution. We provide intuition for the proof of this result in Section 4.3.

4.1. Which constraints matter?

Recall that (Π^*, q^*) is optimal if it is the limit of a sequence $(\Pi_k, q_k)_k$ of price distributions and corresponding worst-case equilibrium quantities whose revenue $R_{q_k}(\Pi_k)$ converges to the seller's optimal value in (P). The next proposition establishes that (Π^*, q^*) can be computed as the solution to an auxiliary program.

[14] See Section 17.2 of Easley and Kleinberg (2010) for a related discussion.

Proposition 2. (Π^*, q^*) *is optimal if and only if it solves*

$$\max_{\Pi \in \Delta(\mathbb{R}_+),\ q \in [0,1]} R_q(\Pi) \tag{P*}$$

$$\textit{subject to} \quad D_{\hat{q}}(\Pi) \geq \hat{q} \quad \forall \hat{q} \in (0, q).$$

Moreover, this program has a maximizer, generating strictly positive revenue.

Proposition 2 elucidates the constraints that are introduced by the seller having a concern for strategic uncertainty. Observe that as in the best-case benchmark of Section 3, program (P*) maximizes over both a price distribution $\Pi \in \Delta(\mathbb{R}_+)$ and an equilibrium quantity $q \in [0,1]$, and any optimum (Π^*, q^*) in (P*) satisfies the equilibrium condition $D_{q^*}(\Pi^*) = q^*$ (for otherwise raising q^* would yield a strict improvement). However, there are additional constraints that (P*) imposes to guarantee that (Π^*, q^*) is optimal under worst-case selection. Plainly, the seller's price distribution cannot admit any lower-quantity equilibrium, and therefore the demand $D_{\hat{q}}(\Pi)$ at any anticipated quantity $\hat{q} < q^*$ must exceed $\hat{q}$. Program (P*) imposes these demand constraints as weak inequalities, with the solution being the limit of a sequence $(\Pi_k, q_k)_k$ that satisfies the constraints strictly for every k.

To prove Proposition 2, we first show that the auxiliary program (P*) is a relaxation of the original program (P). In fact, any price distribution $\Pi \in \Delta(\mathbb{R}_+)$ and its corresponding worst-case equilibrium quantity q are feasible in (P*): if $q = 0$, the program imposes no constraints, and if $q > 0$, then this being the lowest equilibrium quantity under Π implies that the constraints in (P*) hold strictly for all $\hat{q} \in [0, q)$.[15] Next, in the other direction, we show that (P*) cannot yield strictly higher revenue than (P). Given (Π, q) feasible in (P*), we construct a perturbed price distribution Π_ε which coincides with Π except for a small fraction ε of buyers who are offered a zero price. For every $\varepsilon > 0$, the price distribution Π_ε generates a worst-case equilibrium quantity $q_\varepsilon \geq q$. Hence, since revenue is increasing in the quantity demanded and Π_ε

[15] In particular, note that $D_0(\Pi) \geq 0$ and the demand function is continuous in anticipated quantity. Thus, if any $\hat{q} \in (0, q)$ had $D_q(\Pi) \leq q$, then some $\tilde{q} \in [0, \hat{q}]$ would be an equilibrium quantity, meaning q is not the worst equilibrium.

converges to Π as $\varepsilon \to 0$, we obtain $R_{q_\varepsilon}(\Pi_\varepsilon) \geq R_q(\Pi)$ in this limit.

While program (P*) clarifies which constraints are the relevant ones for the seller under worst-case selection, it is not immediate what the solution to this program looks like. The seller chooses a continuum of prices which must satisfy a continuum of demand constraints. At each of these constraints, she faces tradeoffs between increasing one price versus lowering another one to preserve demand, and the tightness of the constraints depends on the relative slopes of the demand function at different anticipated total quantities. It might be intuitive to think that the solution to (P*) should satisfy all the demand constraints with equality—but this may not be feasible for a given target quantity, and even when feasible, we will see that it is not optimal.

In the next two sections, we show that the principle that guides the solution to program (P*) is essentially the same principle behind the seller's solution in the benchmark of best-case selection. This principle is that price dispersion is bad for revenue, so quantity-preserving contractions of the price distribution benefit the seller. Of course, unlike under best-case selection, the result will not be a degenerate price distribution; as discussed in the Introduction, price dispersion is needed to generate strictly positive revenue under worst-case selection. Instead, our analysis will show how this fact can be used to pin down the optimal form of price dispersion.

4.2. Posted price with dispersed discounts

We define a class of functions that we will use in our characterization of the seller's optimal pricing policy.

Definition 1. *Let* $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$ *be right-continuous and nondecreasing. Given* $p \in \mathbb{R}_+$*, say* Γ *is* ***greedy up to p*** *if it satisfies*

$$D_{\hat{q}}(\Gamma) = \hat{q}$$

for all $\hat{q} \in (0, 1)$ *with* $\overline{v}(\hat{q}) \leq p$*. Say* Γ *is* ***greedy*** *if it is greedy up to every* $p \geq 0$*.*

A function Γ that is greedy up to p satisfies the demand constraints in program (P*) with equality for all anticipated quantities for which the highest willingness to pay is no greater than p.[16] This means that Γ iteratively sets to zero the demand-constraint difference $D_{\hat{q}}(\Gamma) - \hat{q}$ starting from the lowest anticipated quantity up to $\underline{q}(p) := \overline{v}^{-1}(p)$. Intuitively, a greedy function follows a greedy procedure: for each anticipated quantity $\hat{q}$ starting from 0, given a measure over prices $[0, \overline{v}(\hat{q}))$, the seller pushes up the next prices as much as possible subject to satisfying the demand constraint at $\hat{q}$. Following this greedy procedure up to q is equivalent to solving the Volterra integral equation $\int_0^{\overline{v}(\hat{q})} \Pi(p) f_{\hat{q}}(p) \, \mathrm{d}p = \hat{q}$ for all $\hat{q} \in (0, q)$.

The next theorem presents our main result.

Theorem 1. *Suppose* (Π^*, q^*) *is optimal, and let* p^* *be the highest price in the support of* Π^**. Then* $p^* \leq p^M(q^*)$*, and* Π^* *is greedy up to* p^**, with a mass point at* p^**.*

A seller's optimal price distribution balances two goals. On the one hand, being concerned with worst-case outcomes, the seller wishes to insulate against low-quantity equilibria. She does so by using a greedy function that seeds the demand from the bottom, placing as little mass on low prices as is needed to iteratively rule out low-demand outcomes.[17] On the other hand, the seller also wishes to extract revenue from the induced higher demand. This is achieved with the mass point at the highest offered price p^*—such an extraction point plays the same role as the seller's posted price $p^M(q^*)$ in the standard monopoly problem with exogenous demand D_{q^*}. The resulting price distribution maximizes the seller's worst-case revenue by minimizing the demand constraints up to anticipated quantity $\underline{q}^* := \overline{v}^{-1}(p^*)$ and satisfying with slack the demand constraints for quantities in $(\underline{q}^*, q^*)$.

Theorem 1 suggests an appealing interpretation for the seller's optimal pricing policy: the seller posts a high price and simultaneously offers personalized

[16] While we require Γ to be right-continuous and nondecreasing, this is actually implied by our greediness condition provided that Γ has bounded variation.

[17] Recall that while the demand constraints in program (P*) are weak inequalities, the solution (Π^*, q^*) is the limit of a sequence $(\Pi_k, q_k)_k$ which, for every k, satisfies the demand constraints strictly and thus rules out equilibrium quantities $\hat{q} < q_k$.

discounts to some buyers to build a high demand. The use of list prices together with promotions and special deals that vary across buyers is common in applications. The shape of Π^* tells us precisely how these personalized discounts are optimally distributed in the population. We show in the Appendix that any greedy function must be continuous and strictly increasing. Hence, it follows from Theorem 1 that the seller's optimal price distribution has only one mass point, and personalized discounts are (fully) dispersed across buyers.

Corollary 1. *Any optimal price distribution is continuous and strictly increasing up to a mass point at the top of its support. Said differently, the seller's policy is a posted price with dispersed discounts.*

In many environments, one can verify directly that the seller's problem admits a unique greedy function Γ^* over $[0, \overline{v}(1))$. In such cases, Theorem 1 reduces the seller's problem to a one-parameter optimization over q^*, as any optimal (Π^*, q^*) must then have Π^* coincide with the unique greedy function Γ^* up to its highest supported price $p^* \in (0, \overline{v}(q^*))$. This is the case in the examples that we describe next, and more generally we show it is always true in the linear demand environment that we focus on in Section 5—see Lemma 8 in the Appendix.

Figure 2 illustrates our results with three examples. Each graph depicts the unique greedy function $\Gamma^*(p)$ (gray dotted line) and the seller's optimal price distribution $\Pi^*(p)$ (black solid line), which is also unique. The first two examples, in the top panel, belong to the linear demand environment—with a willingness-to-pay function $u(\theta, q) = \theta\overline{v}(q)$ and a uniform distribution of types. The first example on the left is the one discussed in the Introduction, with $\overline{v}(q) = q$. The second example on the right takes $\overline{v}(q) = q + q^2$. Finally, the third example in the bottom panel belongs to the proportional values environment, with $u(\theta, q) = \theta q$ and a power distribution of types.

Observe that the first and third examples in Figure 2 both have proportional values, and while they assume different distributions of types, in both cases the unique greedy function is uniform (and given by $\Gamma^*(p) = p/\mathbb{E}[\theta]$). This is not a coincidence, as we report in the next corollary.

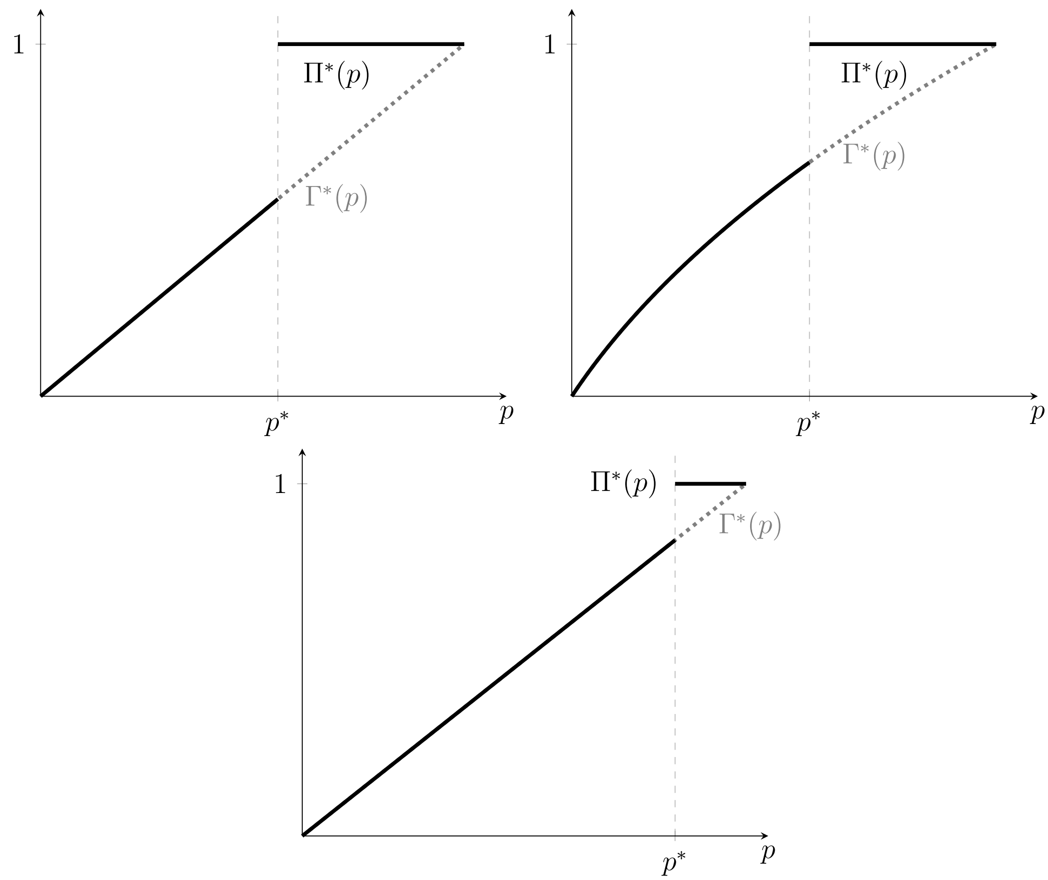

Figure 2: Greedy function (gray dotted line) and optimal price distribution (black solid line). The top-panel examples take a linear demand environment. The left example takes $\overline{v}(q) = q$, and has $\Gamma^*(p) = 2p$ with $p^* \approx 0.28$ and $q^* \approx 0.72$. The right example takes $\overline{v}(q) = q + q^2$, and has $\Gamma^*(p) = (\sqrt{1+4p} - 1)/2 + p/\sqrt{1+4p}$ with $p^* \approx 0.51$ and $q^* \approx 0.76$. The bottom-panel example takes proportional values with $g(\theta) = 2\theta$ over $\Theta = [0, 1]$, and has $\Gamma^*(p) = (3/2)p$ with $p^* \approx 0.56$ and $q^* \approx 0.76$.

Corollary 2. *In the proportional values environment, the seller's policy is a posted price with uniform discounts.*

4.3. Intuition for proof of Theorem 1

We next provide intuition for the proof of Theorem 1. To highlight the main ideas, we focus on the linear demand environment. We comment on the differences with respect to our general proof in the Appendix at the end of this section.

As a preliminary step, we note that under a linear demand, we can rewrite the demand constraints in the auxiliary program (P*) as follows:

$$\int_0^{\overline{v}(\hat{q})} \Pi(p)\mathrm{d}p \geq \hat{q}\overline{v}(\hat{q}) \quad \forall \hat{q} \in (0, q). \tag{2}$$

One can readily verify that all of these constraints are satisfied with equality if Π agrees with the unique greedy function Γ^* up until at least $\overline{v}(q)$, where

$$\Gamma^*(p) = \underline{q}(p) + p\underline{q}'(p). \tag{3}$$

Suppose by contradiction that (Π^*, q^*) is optimal and Π^* is not greedy up to its highest supported price p^*. Since we have shown in Proposition 2 that the seller's optimal value is strictly positive, we take $q^*, p^* > 0$. By definition of the greedy function Γ^*, and assuming here that $\Pi^* - \Gamma^*$ is piecewise monotone, it follows that there exists some price $p' \in [0, \overline{v}(q^*))$ such that $\Pi^*(p) = \Gamma^*(p)$ for $p \leq p'$ and $\Pi^*(p) > \Gamma^*(p)$ right above p'. An illustration is provided in the left panel of Figure 3, where we have drawn $\Gamma^*(p)$ (gray dotted line) for the same environment as in the top left example of Figure 2.

There are two scenarios to consider. First, suppose that there exists a price $p'' \in (p', \overline{v}(q^*))$ such that

$$\int_{p'}^{p''} \left[\Pi^*(p) - \Gamma^*(p)\right] \mathrm{d}p = 0. \tag{4}$$

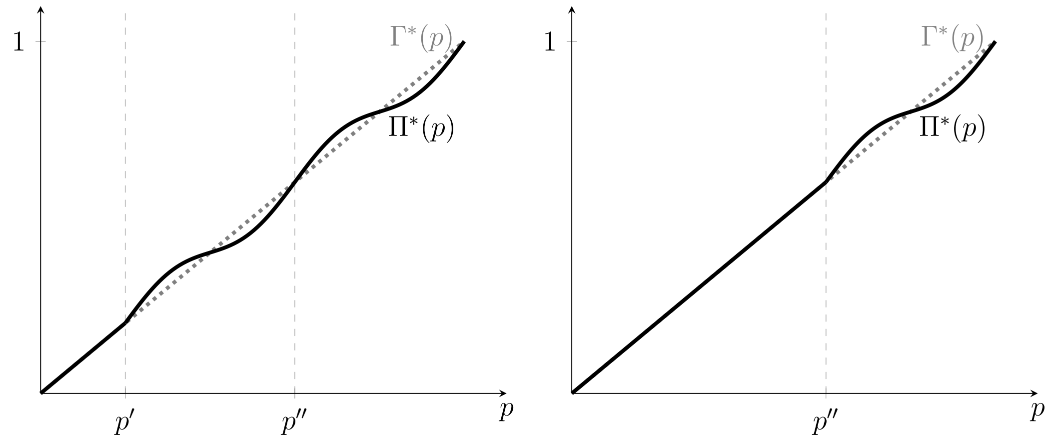

Figure 3: Illustration of arguments in Section 4.3. See the text for details.

Then we can take the lowest such price p'', in which case

$$\int_{p'}^{\hat{p}} [\Pi^*(p) - \Gamma^*(p)] \, \mathrm{d}p > 0 \quad \forall \hat{p} \in (p', p''), \tag{5}$$

as illustrated in Figure 3.

Now let us define a new price distribution $\tilde{\Pi}$ which coincides with the greedy function Γ^* up to p'' and is otherwise equal to Π^*:[18]

$$\tilde{\Pi}(p) = \begin{cases} \Gamma^*(p) & \text{for } p < p'' \\ \Pi^*(p) & \text{otherwise.} \end{cases}$$

The right panel of Figure 3 provides an illustration. By definition of Γ^*, the price distribution $\tilde{\Pi}$ satisfies the demand constraints for all anticipated quantities $\hat{q} \in (0, \underline{q}(p''))$. Moreover, observe that by (4) and (5), $\tilde{\Pi}$ is a mean-preserving contraction of Π^* below p''. MLRP (Assumption 1) therefore implies $D_{\hat{q}}(\tilde{\Pi}) \geq D_{\hat{q}}(\Pi^*)$ for all $\hat{q} \in [\underline{q}(p''), 1]$, which means that $\tilde{\Pi}$ also satisfies the demand constraints for all anticipated quantities $\hat{q} \in [\underline{q}(p''), q^*)$. Furthermore, cross regularity (Assumption 3) implies $R_{\hat{q}}(\tilde{\Pi}) > R_{\hat{q}}(\Pi^*)$ for all

[18] We can verify that the price p'' satisfies $p'' < p^*$ and $\Gamma^*(p'') \leq \Pi^*(p'')$, so $\tilde{\Pi}$ is a distribution function.

$\hat{q} \in [\underline{q}(p''), 1]$. It follows that $\tilde{\Pi}$ yields strictly higher revenue than Π^*, contradicting the assumption that Π^* is optimal.

We are then left with the second scenario, in which no $p'' \in (p', \overline{v}(q^*))$ exists that satisfies equation (4). In this case, each $\hat{p} \in (p', \overline{v}(q^*))$ has

$$\int_{p'}^{\hat{p}} [\Pi^*(p) - \Gamma^*(p)] \, \mathrm{d}p > 0.$$

By (2), it follows that the corresponding demand constraints are satisfied with slack; that is, $D_{\hat{q}}(\Pi^*) > \hat{q}$ for all anticipated quantities $\hat{q} \in (\underline{q}(p'), q^*)$. However, this means that if Π^* puts any mass (strictly) above p', then again a strict improvement is feasible. Specifically, if Π^* is supported on more than one price above p', we show that a small mean-preserving contraction above p' preserves the demand constraints (by them being slack) and increases revenue (by cross regularity). If Π^* has only one mass point above p', then satisfaction of the demand constraints for quantities right above $\underline{q}(p')$ requires a mass point also at p', and we show that revenue can be increased with a small contraction that takes mass from these two points. We thus conclude that Π^* cannot have support above p', i.e., $p' =: p^*$. This contradicts the assumption that Π^* is not greedy up to p^*.

The steps above yield that any optimal price distribution Π^* coincides with the greedy function Γ^* up to its highest supported price p^*. Theorem 1 also states that p^* is no greater than the monopoly price $p^M(q^*)$. This is intuitive: by definition of $p^M(q^*)$, any price distribution with highest price $p^* > p^M(q^*)$ can be improved upon by lowering all prices above $p^M(q^*)$ to this level. Finally, to prove that Π^* has a mass point at p^*, observe that since $p^M(q^*) < \overline{v}(q^*)$, we have $p^* < \overline{v}(q^*)$. Hence, while the greedy function Γ^* up to p^* satisfies the demand constraints up to $\underline{q}^* = \underline{q}(p^*)$, there must be a mass point at p^* to satisfy the demand constraints over $(\underline{q}^*, q^*)$.

The proof of Theorem 1 in the Appendix proceeds via perturbations as we did here: taking a price distribution Π^* that is not greedy up to the top of its support, and showing how it can be improved while preserving the demand constraints. However, we do not build on a fixed greedy function Γ^*, nor do we rely on $\Pi^* - \Gamma^*$ being well-behaved. Instead, we show that given Π^*, we can locate an interval of anticipated quantities where the demand constraints are slack, and where we can apply contraction arguments analogous to those used in the second scenario above. For locating such an interval, concave externalities (Assumption 2) is important. For arguing that contractions improve revenue while satisfying the demand constraints, a difficulty is that these constraints do not take the form of majorization as in (2) outside the linear demand environment. While this means that we cannot use off-the-shelf comparative statics on mean-preserving contractions as we did above, we show that similar comparative statics can be derived for our general model. This step extends results from Rappoport (2024); see Lemma 3 in the Appendix.

5. The effects of externalities

We use our characterization of the seller's optimal pricing policy to study the effects of externalities. In Section 5.1, we compare the seller's solution under worst-case selection to the best-case benchmark of Section 3. In Section 5.2, we examine how the solution changes as the externalities in buyers' purchasing decisions become stronger. Finally, in Section 5.3, we consider a setting where buyers belong to groups with heterogeneous strength of externalities, and the seller's price offers can condition on buyer group. Throughout this section, we focus on the linear demand environment.

5.1. Worst-case versus best-case

The seller's optimal pricing policy in Theorem 1 is shaped by her concern for strategic uncertainty. Recall from Proposition 1 that under best-case selection—i.e., if the seller could choose the equilibrium that buyers play given her price offers—a posted price mechanism would be optimal. Instead, when concerned with worst-case outcomes, Theorem 1 says that the seller uses a posted price together with personalized discounts. An immediate consequence of the seller's worst-case focus is thus price dispersion. But, what does this imply for price levels and for the induced total quantity of trade? And how does the resulting consumer surplus compare under worst-case versus best-case selection? The next proposition provides answers to these questions. We denote the consumer surplus associated with anticipated quantity $q \in [0,1]$ and price $p \in \mathbb{R}_+$ by

$$\mathrm{CS}_q(p) := \int_p^{\overline{v}(q)} D_q(v)\ \mathrm{d}v,$$

and let $\mathrm{CS}_q(\Pi) := \int \mathrm{CS}_q(p)\ \mathrm{d}\Pi(p)$ for any price distribution $\Pi \in \Delta(\mathbb{R}_+)$.

Proposition 3. *Take the linear demand environment. Relative to the best-case benchmark, the seller's worst-case solution has a higher maximum offered price $p^* > p^B$, induces a higher total quantity $q^* > q^B$, and yields a higher consumer surplus $CS_{q^*}(\Pi^*) > CS_{q^B}(p^B)$.*

This result reveals that not all buyers benefit from lower prices when the seller is concerned with worst-case outcomes. Some buyers receive generous discounts as the seller seeks to ensure a high demand, but others receive price offers strictly higher than the seller's best-case posted price. At the same time, Proposition 3 tells us that buyers on average do purchase at a lower price in the worst-case solution, and thus the total quantity demanded is higher than in the best-case benchmark. Interestingly, while the seller is concerned with ruling out low-quantity equilibria—and must therefore offer discounts and sacrifice revenue to ensure any quantity as a worst-case equilibrium—she ends up inducing a higher quantity of trade than in the absence of this concern. Proposition 3 further establishes that, as a consequence, consumer surplus increases due to the seller's worst-case focus.

The linear-demand example discussed in the Introduction (with $\overline{v}(q) = q$) provides an illustration of the comparisons reported in Proposition 3. As noted, the seller's worst-case and best-case solutions in that setting have maximum prices $p^* \approx 0.28 > 0.22 \approx p^B$ and total quantities $q^* \approx 0.72 > 0.66 \approx q^B$. The resulting consumer surpluses are $\mathrm{CS}_{q^*}(\Pi^*) \approx 0.19 > 0.15 \approx \mathrm{CS}_{q^B}(p^B)$.

To prove Proposition 3, we use our characterizations of the seller's worst-case and best-case optima. By Theorem 1, any optimal worst-case price distribution is a function $\Pi(\cdot|\hat{p})$ that coincides with the greedy function—unique in the linear demand environment—up to a highest supported price $\hat{p}$. Define $\mathcal{R}(\hat{p}, \hat{q})$ as the seller's worst-case revenue given such a price distribution

$\Pi(\cdot|\hat{p})$ and a buyers' anticipated quantity $\hat{q}$. In a worst-case equilibrium, $\hat{q}$ is equal to the lowest quantity demanded given that $\Pi(\cdot|\hat{p})$ is the limit worst-case price distribution; call it $\mathcal{Q}(\hat{p})$. Then $\mathcal{R}(\hat{p}, \mathcal{Q}(\hat{p}))$ gives the seller's worst-case revenue parametrized by $\hat{p}$. We define analogous objects for the best-case problem, with $\mathcal{R}^B(\hat{p}, \hat{q})$ being the seller's best-case revenue given a posted price $\hat{p}$ and anticipated quantity $\hat{q}$, and $\mathcal{R}^B(\hat{p}, \mathcal{Q}^B(\hat{p}))$ taking $\hat{q}$ to equal the highest equilibrium quantity $\mathcal{Q}^B(\hat{p})$ under $\hat{p}$.

Our analysis is facilitated by the fact that, in the linear demand environment, these worst-case and best-case revenue functions are strictly quasiconcave, with unique interior optima p^* and p^B given by

$$\left.\frac{\mathrm{d}\mathcal{R}(\hat{p}, \mathcal{Q}(\hat{p}))}{\mathrm{d}\hat{p}}\right|_{\hat{p}=p^*} = 0 \quad \text{and} \quad \left.\frac{\mathrm{d}\mathcal{R}^B(\hat{p}, \mathcal{Q}^B(\hat{p}))}{\mathrm{d}\hat{p}}\right|_{\hat{p}=p^B} = 0. \tag{6}$$

Hence, to establish the ranking between p^* and p^B, it suffices to sign

$$\frac{\mathrm{d}\mathcal{R}}{\mathrm{d}\hat{p}} = \underbrace{\frac{\partial \mathcal{R}}{\partial \hat{p}}}_{\substack{\text{monopoly} \\ \text{effect}}} + \underbrace{\frac{\partial \mathcal{R}}{\partial \hat{q}} \frac{\mathrm{d}\mathcal{Q}}{\mathrm{d}\hat{p}}}_{\substack{\text{externality} \\ \text{effect}}} \tag{7}$$

at $\hat{p} = p^B$. We call the first term on the right-hand side the **monopoly effect**. This effect tells us how revenue changes with $\hat{p}$ while keeping the anticipated quantity $\hat{q}$, and thus the demand function $D_{\hat{q}}$, fixed. As is familiar, raising $\hat{p}$ increases revenue from inframarginal buyers via a higher price, but reduces revenue from marginal buyers via a lower quantity. Note that if $\hat{p} = p^B$, the quantity demanded at the worst-case highest price $\hat{p}$ is larger than the quantity demanded at the best-case posted price p^B. A comparison of the monopoly effects then follows from (cross) regularity (Assumption 3): conditional on pricing at $\hat{p}$, the worst-case monopoly effect of raising $\hat{p}$ starting from $\hat{p} = p^B$ is higher (i.e., more positive) than the analog best-case monopoly effect.

The second term on the right-hand side of (7) is the **externality effect**. This effect tells us how revenue changes as the demand function $D_{\hat{q}}$ shifts towards the new equilibrium—that is, given that the anticipated quantity $\hat{q}$ must adjust to match the quantity demanded $\mathcal{Q}(\hat{p})$ following an increase in $\hat{p}$.

We show that conditional on pricing at $\hat{p}$, the worst-case externality effect of raising $\hat{p}$ starting from $\hat{p} = p^B$ is higher than the analog best-case externality effect. Hence, given the definition of p^B in (6), the monopoly and externality effects imply $\mathrm{d}\mathcal{R}(\hat{p}, \mathcal{Q}(\hat{p}))/\mathrm{d}\hat{p} > 0$ at $\hat{p} = p^B$. We conclude that the worst-case highest price p^* is strictly higher than the best-case posted price p^B.

The idea behind the ranking of the worst-case and best-case quantities, q^* and q^B, is similar. We show that $\mathcal{R}(\hat{p}, \mathcal{Q}(\hat{p}))$ increases as $\hat{p}$ is reduced from a level that makes the worst-case equilibrium quantity equal to q^B, and therefore the optimal such quantity must satisfy $q^* > q^B$. It is worth noting that this finding contrasts with results in complete-information settings. As Segal (2003) shows, under complete information and general conditions, a robustness requirement in the form of worst-case or unique implementation would only lower the total quantity of trade relative to best-case implementation.

Finally, the quantity result allows us to prove the ranking on consumer surplus. In particular, using $q^* > q^B$, we show that the average price offered under a price distribution $\Pi(\cdot|\overline{v}(q^B))$ is no greater than p^B.[19] Since consumer surplus is decreasing in average price, and is strictly increasing in price dispersion due to buyer option value, it follows that $\mathrm{CS}_{q^B}(\Pi(\cdot|\overline{v}(q^B))) > \mathrm{CS}_{q^B}(p^B)$. Moreover, since consumer surplus is also increasing in quantity, we obtain

$$\mathrm{CS}_{q^*}(\Pi(\cdot|p^*)) \geq \mathrm{CS}_{q^B}(\Pi(\cdot|p^*)) = \mathrm{CS}_{q^B}(\Pi(\cdot|\overline{v}(q^B))) > \mathrm{CS}_{q^B}(p^B).$$

5.2. Strength of externalities

The externalities in buyers' purchasing decisions are a key novel ingredient of our seller's problem. We next study how the seller's solution in Theorem 1 changes as these externalities become stronger. Recall that in the linear demand environment, a buyer's willingness to pay given type $\theta \in \Theta$ and anticipated quantity $q \in [0, 1]$ is $u(\theta, q) = \theta\overline{v}(q)$. The strength of externalities is reflected in the function $\overline{v}$.

[19] This follows from the fact that, given $q^* > q^B$, the demand constraint in program (P*) is satisfied at anticipated quantity $\hat{q} = q^B$.

Definition 2. *In the linear demand environment, say* $\overline{v}_1$ *has* **stronger externalities** *than* $\overline{v}_0$ *if, for all* $q \in (0, 1]$,

(i) $\overline{v}_1(q) > \overline{v}_0(q)$, *and*

(ii) $\overline{v}_1(q)/\overline{v}_0(q)$ *is strictly increasing in* q.

Intuitively, buyers' purchasing decisions are more complementary if their willingness to pay grows with the anticipated total quantity demanded at a higher rate, as captured by (ii).[20] Since our model assumes $\overline{v}(0) = 0$, it is then also natural that buyers' willingness to pay under any given anticipated total quantity will be higher when externalities are stronger, as captured by (i).

Proposition 4. *Take the linear demand environment. Suppose* $\overline{v}_1$ *has stronger externalities than* $\overline{v}_0$*, with corresponding optimal price distributions* Π_1^* *and* Π_0^**. Relative to* Π_0^**, then* Π_1^* *induces a higher total quantity* $q_1^* > q_0^*$*. Moreover,* Π_1^* *puts lower weight on low prices:* $\Pi_1^*(p) < \Pi_0^*(p)$ *for all* $p < \min\{p_1^*, p_0^*\}$.

An increase in the strength of externalities makes it less costly for the seller to insulate against low-demand equilibria. Specifically, take any anticipated quantity $\hat{q} \in (0, 1)$ and greedy prices over $[0, \overline{v}(\hat{q}))$ that satisfy the demand constraints in program (P*) up to $\hat{q}$. Under stronger externalities, because the demand is more responsive to anticipated quantity, the seller can then satisfy the demand constraint at $\hat{q}$ without the need to offer price $\overline{v}(\hat{q})$ to such a large mass of buyers. As a result, the optimal price distribution places relatively less weight on discounted prices below a given posted price. Moreover, because the seller can guarantee a given equilibrium quantity while charging higher prices, it is optimal for her to induce a higher quantity when externalities are stronger. These price and quantity effects combined explain why Proposition 4 does not pin down how the posted price itself changes with the externalities.

For illustration, we can compare the linear-demand examples shown in the top panel of Figure 2. The second example on the right (with $\overline{v}(q) = q + q^2$)

[20] Observe that taking $\overline{v}_1'(\cdot) > \overline{v}_0'(\cdot)$ would not yield the desired externality order: multiplying $\overline{v}$ by a constant $\kappa > 0$ has no effect on the seller's solution up to a change of numeraire.

has stronger externalities than the first example on the left (with $\overline{v}(q) = q$), and accordingly induces a higher total quantity of trade (as reported in the figure caption). The second example also exhibits a higher posted price and lower weight on discounted prices below the first-example posted price.

The proof of the comparative static concerning the seller's optimal price distribution follows directly from equation (3), which defines the unique greedy function in the linear demand environment. If $\overline{v}_1$ has stronger externalities than $\overline{v}_0$, then the greedy function under $\overline{v}_1$ is lower than that under $\overline{v}_0$ in the first-order stochastic dominance (FOSD) sense.

To prove the comparative static on the optimal total quantity, we use arguments similar to those described in Section 5.1. Given $\overline{v}_1$ and $\overline{v}_0$, let $\mathcal{R}_1(\hat{p}, \mathcal{Q}_1(\hat{p}))$ and $\mathcal{R}_0(\hat{p}, \mathcal{Q}_0(\hat{p}))$ be the respective revenue functions parametrized by the highest offered price $\hat{p}$. We study how the strong-externality revenue $\mathcal{R}_1$ changes as we increase the highest price $\hat{p}$, starting from a level that makes the induced strong-externality quantity equal to the optimal weak-externality quantity q_0^*. By the FOSD ranking of the greedy functions, such a starting level for $\hat{p}$ is strictly higher than p_0^*. We then show that increasing $\hat{p}$ from that level causes strong-externality monopoly and externality effects, as defined in equation (7), which are both lower (i.e., more negative) than the corresponding weak-externality effects caused by increasing $\hat{p}$ from p_0^*.[21] Since the latter weak-externality effects add to zero by definition of p_0^*, this means that the strong-externality revenue $\mathcal{R}_1$ can be increased by lowering $\hat{p}$. Thus, we obtain that the optimal total quantities satisfy $q_1^* > q_0^*$, as stated in Proposition 4.

5.3. Heterogeneity

In the previous section, we studied how the seller's pricing policy changes as the externalities in buyers' purchasing decisions become stronger. A related but distinct question is how the seller's policy changes if the strength of externalities varies across buyers. For example, take a seller of file sharing services.

[21] The monopoly effect is lower under stronger externalities because marginal buyers are located at a higher price point, so the loss in revenue from their quantity going down is more pronounced. The externality effect is lower because the feedback effects of a lower quantity are larger under stronger externalities.

Because these services are more heavily used in the corporate sector, corporate buyers' willingness to pay would tend to be higher and to grow at a faster rate with the total number of subscribers compared to that of retail buyers. How should the seller's price offers take this heterogeneity into account?

We consider $N > 1$ buyer groups indexed by $n \in \{1, \dots, N\}$, each making up a proportion $\lambda_n > 0$ of the population, with $\sum_n \lambda_n = 1$. A buyer's willingness to pay is increasing in the anticipated quantity q demanded by all buyers, but this externality is stronger on buyers in higher-indexed groups. Specifically, in the linear demand environment with $u(\theta, q) = \theta\overline{v}(q)$, and consistent with Definition 2, we assume that for all $q \in [0, 1]$ and all $n \in \{1, \dots, N-1\}$: *(i)* $\overline{v}_{n+1}(q) > \overline{v}_n(q)$, and *(ii)* $\overline{v}_{n+1}(q)/\overline{v}_n(q)$ is strictly increasing in q.

The seller's price offers can condition on both a buyer's identity i and the group n to which the buyer belongs (but not on the buyer's private type θ). The seller's problem thus amounts to choosing a price distribution $\Pi_n \in \Delta(\mathbb{R}_+)$ for each buyer group $n \in \{1, \dots, N\}$, with the objective of maximizing her total worst-case revenue. Given an anticipated total quantity $q \in [0, 1]$ and a price $p \in \mathbb{R}_+$, denote the (unweighted) quantity demanded by group-n buyers by $D_{n,q}(p) := 1 - p/\overline{v}_n(q)$, and let $D_{n,q}(\Pi_n) := \int D_{n,q}(p)\ \mathrm{d}\Pi_n(p)$ and $R_{n,q}(\Pi_n) := \int pD_{n,q}(p)\ \mathrm{d}\Pi_n(p)$ for any $\Pi_n \in \Delta(\mathbb{R}_+)$. Applying the logic of Proposition 2, we can write the seller's problem analogously as we did in program (P*) for our baseline model:

$$\max_{\{\Pi_n \in \Delta(\mathbb{R}_+)\}_n, q \in [0,1]} \sum_n \lambda_n R_{n,q}(\Pi_n) \tag{P^*_N}$$

$$\text{subject to} \quad \sum_n \lambda_n D_{n,\hat{q}}(\Pi_n) \geq \hat{q} \quad \forall \hat{q} \in (0, q).$$

As in (P*), the demand constraints in (P^*_N) say that to implement an equilibrium total quantity $q \in [0, 1]$, the seller must rule out any lower quantity as an equilibrium. That requires that for each anticipated quantity $\hat{q} \in (0, q)$, the total quantity demanded exceed $\hat{q}$.[22] Now, in this setting, the total quantity

[22] As in (P*), we impose the demand constraints as weak inequalities, with the solution being the limit of a sequence $(\{\Pi_{nk}\}_n, q_k)_k$ that satisfies them strictly for every k.

demanded is the sum of the demands from each of the N buyer groups. The seller thus makes a choice on how to use the different groups to build the demand up to q. The following definition introduces a class of price distributions that build the demand in an ordered manner. We let $\underline{q}_n := \overline{v}_n^{-1}$.

Definition 3. *Given prices $p_1 \leq \cdots \leq p_N$, say price distributions $(\Pi_n)_{n=1}^N$ are* ***residual greedy up to*** $\boldsymbol{(p_n)_{n=1}^N}$ *if the following is true for each $n \in \{1, \ldots, N\}$:*

- *if $p_n \leq \overline{v}_n(q_{n-1})$, then Π_n is degenerate on p_n,*
- *if $p_n > \overline{v}_n(q_{n-1})$, then $\operatorname{supp}(\Pi_n) = [\overline{v}_n(q_{n-1}), p_n]$ and*

$$\sum_{m=1}^{n} \lambda_m D_{m,\hat{q}}(\Pi_m) = \hat{q} \quad \forall \hat{q} \in (q_{n-1}, \underline{q}_n(p_n)),$$

where $q_n := \max\left\{q \in [0,1] : \sum_{m=1}^{n} \lambda_m D_{m,q}(\Pi_m) = q\right\}$.

Price distributions $(\Pi_n)_n$ that are residual greedy up to $(p_n)_n$ have two key properties. First, since the quantities $(q_n)_n$ as defined must satisfy $p_n \leq \overline{v}_n(q_n)$ and $q_1 < q_2 < \ldots < q_N$, the supports of the price distributions are ordered.[23] This means that all buyers in group n are offered lower prices than any buyer in group $n+1$, and the seller uses buyers only from groups $\{1, \ldots, n\}$ to satisfy the demand constraints up to anticipated quantity q_n. Second, given the price distributions for groups $\{1, \ldots, n-1\}$, the price distribution for group n makes the demand constraints for anticipated quantities $\hat{q} \in (q_{n-1}, \underline{q}_n(p_n))$ hold with equality. Intuitively, the seller follows a greedy procedure as in our main model, but because these demand constraints aggregate the quantity demanded by buyers in all groups $(1, \ldots, n)$, the prices are greedy in a residual sense: the seller offers discounts to group-n buyers only as much as is needed to build the residual demand not fulfilled by lower-index-group buyers.

We show that the seller's optimal pricing policy consists of price distributions that are residual greedy up to their highest supported prices.

[23] Observe that there is no circularity in Definition 3 since Π_n depends on $(q_1, \ldots, q_{n-1})$ while q_n depends on Π_n.

Proposition 5. *Take the linear demand environment with buyer groups* $n \in \{1, \ldots, N\}$ *indexed by increasing strength of externalities. Suppose* $((\Pi_n^*)_{n=1}^N, q^*)$ *is optimal, and let* p_n^* *be the highest price in the support of* Π_n^**. Then the price distributions* $(\Pi_n^*)_{n=1}^N$ *are residual greedy up to* $(p_n^*)_{n=1}^N$*, and* Π_N^* *has a mass point at* $p_N^* < \overline{v}_N(q^*)$*. Therefore, for each* $n \in \{1, \ldots, N-1\}$*,*

$$\max \operatorname{supp}(\Pi_n^*) \leq \min \operatorname{supp}(\Pi_{n+1}^*).$$

Moreover, this inequality is strict unless $p_n^* = p_{n+1}^* = 0$*.*

This result sheds light on how the seller optimally builds the demand towards an equilibrium total quantity. Buyers from strong-externality groups are more responsive to the anticipated quantity of trade than those from weak-externality groups. Hence, the seller benefits from offering lower prices to weak-externality buyers in order to provide assurance of a higher total quantity to strong-externality buyers; this allows her to extract higher revenue from the latter. Going back to the example of corporate and retail buyers of file sharing services, Proposition 5 says that all retail buyers will enjoy larger discounts than any corporate buyer.

The proposition further shows that the methodology from our main model extends to the setting with heterogeneous buyer groups. Once we establish that the optimal price distributions $(\Pi_n^*)_n$ have ordered supports—more specifically, that any prices $(p_n)_n$ respectively in the supports of $(\Pi_n^*)_n$ have $\underline{q}_n(p_n) \leq \underline{q}_{n+1}(p_{n+1})$—then we are able to apply the arguments of Theorem 1 to each of the N buyer groups. This yields the characterization in Proposition 5, with price distributions that are residual greedy up to the top of their supports, and a mass point at p_N^*, as well as possibly mass at other points in $(p_1^*, \ldots, p_{N-1}^*)$.[24] The interpretation is intuitive: the seller sets a high posted price together with group-exclusive discounts and personalized discounts within each group.

Figure 4 provides an illustration. We take a population with $N = 2$ buyer

[24] Observe that residual greediness can be consistent with a price distribution Π_n^* also having a mass point at the lower limit of its support. An example is shown in Figure 4.

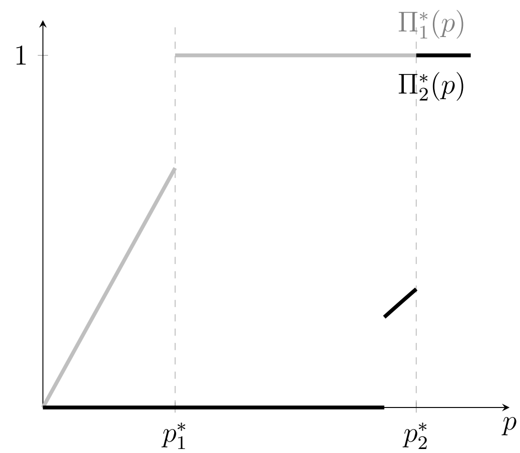

Figure 4: Optimal price distributions for a population with 2 buyer groups with equal weights, where group-1 and group-2 buyers have willingness to pay as in the first and second examples of Figure 2 respectively. For group 1 (light gray line), we obtain $\Pi_1^*(p) = 4p$ for $p < p_1^*$ and $\Pi_1^*(p) = 1$ for $p \geq p_1^*$, with $p_1^* \approx 0.17$ and $q_1 \approx 0.33$. For group 2 (black line), we obtain $\Pi_2^*(p) = 0$ for $p < \overline{v}_2(q_1)$, $\Pi_2^*(p) = (1 + 6p - 2\sqrt{1+4p} + p_1^*(1 - 2p_1^*))/\sqrt{1+4p}$ for $\overline{v}_2(q_1) \leq p < p_2^*$, and $\Pi_2^*(p) = 1$ for $p \geq p_2^*$, where $p_2^* \approx 0.48$ and $q_2 = q^* \approx 0.74$.

groups with equal weights. Group-1 and group-2 buyers have willingness to pay as given respectively in the first and second examples of Figure 2. We can interpret the seller's solution as posting a price of p_2^* and offering all buyers in the weak-externality group 1 a group-exclusive discount of $p_2^* - p_1^*$, in addition to offering personalized discounts to some buyers in this group and some buyers in the strong-externality group 2. In this way, the seller builds the demand with group-1 buyers up to a quantity q_1, and then extracts higher revenue from group-2 buyers as she continues to grow the demand with buyers from both groups up to the equilibrium quantity $q^* > q_1$.

6. Discussion

In this section, we describe different variants of our model, discuss how our analysis and results would (or not) change, and offer some concluding remarks.

Complete information. Our seller's problem has two key features: externalities in consumption and unobservable buyer types. In Section 3, we studied a benchmark describing what happens when either the externalities are absent or the strategic uncertainty they generate is not a concern for the seller. That benchmark placed our analysis within the literature on monopoly pricing. We next consider the other natural benchmark for our problem, in which the externalities and the concern for strategic uncertainty are present, but buyer types are observable. This benchmark connects our analysis to the literature on contracting with externalities, which until now had focused on complete-information settings.

Take our baseline model but suppose the seller can make price offers that condition on both a buyer's identity i and his type θ. Since the seller knows exactly how much each buyer (i, θ) is willing to pay for each anticipated total quantity, her problem simplifies significantly. Given the increasing externalities, it is easy to see that the seller will want to ensure that all buyers purchase. This means that all buyers purchasing must be the unique equilibrium, and thus the unique rationalizable outcome. Therefore, the seller's problem reduces to choosing an order in which buyers iteratively delete the no-purchase action as being dominated, together with revenue-maximizing price offers that implement this iterated deletion. The solution prescribes a permutation of buyers, such that each buyer in the permutation is indifferent over purchasing if all buyers preceding him purchase and the rest do not.[25] If $u(\theta, q)$ is supermodular (as in our linear-demand and proportional-values environments), then an optimal permutation orders buyers in increasing type order.[26]

This approach is the same as used in other papers on contracting with externalities. However, this methodology is not available to us in our model with incomplete information. Plainly, the fact that types are unobservable means that the seller cannot control the order in which buyers iteratively delete the no-purchase action. Our analysis develops a new methodology that

[25] Recall that we have assumed that buyers purchase under indifference.

[26] The intuition for this order is similar to that in Section 5.3: the seller benefits from building the demand with lower types so that she can extract more value from higher types.

excludes low-revenue outcomes by working not through the buyer types but through the anticipated quantities of trade. The seller's solution iteratively deletes anticipated quantities as candidates for equilibrium quantities. Observe that, in this solution, the order in which buyers delete the no-purchase action in not necessarily monotonic, neither in their types nor in their price offers.

In addition to requiring a new methodology, our incomplete-information problem yields results that are qualitatively different from those obtained under complete information. As noted, when types are observable, the seller induces the whole population of buyers to purchase. Moreover, except in special cases, no two buyers receive the same price offer.[27] Instead, our model with unobservable types yields exclusion and comparative statics on the total quantity of trade, as well as the result that any optimal price distribution is continuous and strictly increasing up to a mass point at the top. The latter allows us to interpret the seller's solution as a posted price with dispersed discounts, and thus to relate our findings to pricing policies used in applications.

Screening menus. We have phrased our model with the seller choosing personalized price offers. Since buyers have private information about their payoff types, it is natural to ask whether the seller could do better with more sophisticated mechanisms. In this section, we argue that our focus on price offers is without loss of generality within the class of public bilateral contracts.

Let $\mathcal{M}$ denote the set of all compact subsets of $[0,1]\times\mathbb{R}_+$ that contain $(0,0)$. We consider a general contracting environment in which the seller offers a menu $M_i \in \mathcal{M}$ to each buyer $(i,\theta) \in I \times \Theta$, and buyers then simultaneously choose an option from their offered menus. Each menu option specifies a probability of trade $x \in [0,1]$ and a transfer $t \in \mathbb{R}_+$ from the buyer to the seller, with $(0,0)$ corresponding to a buyer's option of not purchasing the good and not making any transfer. Clearly, this is a generalization of our main model, as menus in $\mathcal{M}^P := \{\{(0,0),(1,p)\} : \ p \in \mathbb{R}_+\}$ correspond exactly to price offers.

For any menu $M \in \mathcal{M}$ and willingness to pay $v \in \mathbb{R}_+$, let $(x_M(v), t_M(v))$

[27] This is always true if $u(\theta,q)$ is supermodular. In our Introduction example, where $u(\theta,q) = \theta q$ and types are drawn uniformly from $\Theta = [0,1]$, the optimal price distribution under complete information is $\Pi^C(p) = \sqrt{p}$ for $p \in [0,1]$.

be the element of $\arg\max_{(x,t)\in M}(xv - t)$ with highest x. If a buyer anticipates total quantity of trade q and faces menu offer M, his expected quantity demanded is $D_q(M) := \int_0^{\overline{v}(q)} x_M(v) f_q(v)\ \mathrm{d}v$, and the expected revenue he generates is $R_q(M) := \int_0^{\overline{v}(q)} t_M(v) f_q(v)\ \mathrm{d}v$.[28] Analogous to our main model, we can summarize the seller's mechanism choice via a distribution—here, a distribution $\mu \in \Delta\mathcal{M}$ over menu offers. Given such a μ, a total quantity q is an equilibrium quantity if and only if $q = D_q(\mu) := \int D_q(M)\ \mathrm{d}\mu(M)$, and the resulting revenue is $R_q(\mu) := \int R_q(M)\ \mathrm{d}\mu(M)$.

We argue that any menu distribution $\mu \in \Delta\mathcal{M}$ admits some price distribution $\Pi_\mu \in \Delta(\mathbb{R}_+)$ with the same set of equilibrium quantities $q \in [0, 1]$ and generating the same revenue for every equilibrium quantity. The idea is simple. First, it follows by standard arguments (Myerson, 1981) that any menu $M \in \mathcal{M}$ can be replaced by a revenue-equivalent random posted price. That is, given M, we can define a distribution Π_M such that a buyer who has willingness to pay $v \in [0, \overline{v}(1)]$ and faces a random posted price with distribution Π_M (and purchases whenever doing so is weakly optimal) would then purchase with probability $x_M(v)$ and generate an expected transfer of $t_M(v)$.[29] Thus, for any $q \in [0, 1]$, we obtain $D_q(\Pi_M) = D_q(M)$ and $R_q(\Pi_M) = R_q(M)$. Next, because there is a continuum of buyers, we can take the distribution of prices that the individual random posted prices generate in the population and implement it directly as a distribution of price offers. That is, we can define Π_μ to be the barycenter $\int \Pi_M\ \mathrm{d}\mu(M)$, yielding $D_q(\Pi_\mu) = D_q(\mu)$ and $R_q(\Pi_\mu) = R_q(\mu)$ for every $q \in [0, 1]$.

The implication is that our focus on price offers rather than menu offers is without loss. Instead, what matters for our analysis is our maintained assumption that contracts are bilateral and public. Bilateral contracts means that the contract offered to a buyer cannot directly condition on the purchasing decisions of other buyers. If such multilateral contract offers were feasible, they could mitigate the seller's concern for strategic uncertainty.[30] Multilateral

[28] The assumption that buyers choose the highest-x option among their preferred menu options corresponds to our main model's assumption that buyers purchase when indifferent.

[29] Taking $\nu > 0$, we can let $\Pi_M(v) = x_M(v)$ for $v < \overline{v}(1) + \nu$, and $\Pi_M(v) = 1$ otherwise.

[30] For instance, the seller would be able to guarantee the best-case pricing outcome by

contracts are often difficult to verify and enforce in practice, and for this reason they are commonly ruled out in the contracting-with-externalities literature.[31] Finally, public contracts means that buyers know the realized distribution of prices offered in the population. Whether revenue can be improved in a setting with private contracts—as is the case in the moral-hazard problem of Halac et al. (2021)—is an open question.

Warm start. Our model assumes a cold-start problem: no buyer is willing to purchase at a strictly positive price if he anticipates that no other buyer will purchase. Formally, we assumed that the highest willingness to pay as a function of the anticipated quantity of trade, $\overline{v}(q)$, satisfies $\overline{v}(0) = 0$. In this section, we discuss how our results change if we relax this assumption and consider a "warm-start" model where $\overline{v}(0) > 0$. We maintain all of our other assumptions, including that $\overline{v}$ is continuously differentiable with $\overline{v}' > 0$.

Conceptually, our analysis can be extended to the warm-start model with little modification. Both our restatement of the seller's problem in Proposition 2 and our characterization of the seller's solution in Theorem 1 continue to apply. The latter in particular says that any optimal (Π^*, q^*) has Π^* greedy up to its highest supported price p^*, and thus that our definition of greediness remains useful for describing the seller's optimal price distribution. A key difference, however, is that greediness now implies zero mass on prices strictly below $\overline{v}(0)$, so we must have $\Pi^*(p) = 0$ for all $p < \min\{p^*, \overline{v}(0)\}$. Therefore, Theorem 1 in the warm-start model says that Π^* takes one of two forms: either Π^* has no supported prices strictly below $\overline{v}(0)$—in which case it takes the form of a posted price with dispersed discounts, as in our cold-start model—or Π^* is degenerate on $p^* < \overline{v}(0)$—in which case it is simply a posted price.

The intuition for why a degenerate price distribution could be optimal for the seller under warm start can be seen immediately by taking $\overline{v}(0)$ to be high enough. Indeed, observe that by (the application of) Theorem 1, any optimal (Π^*, q^*) has Π^* with highest supported price $p^* \leq p^M(q^*)$, where $p^M(q^*)$ is

offering each buyer a contract that specifies the best-case price p^B conditional on total quantity $q \geq q^B$, and a zero price otherwise.

[31] See, e.g., Innes and Sexton (1994), Segal (2003), and Halac et al. (2020).

the monopoly price that obtains under an exogenous demand D_{q^*}. Hence, a sufficient condition for the seller to choose a degenerate price distribution in the warm-start model is $\overline{v}(0) > p^M(1)$. In this case, the externalities in consumption operate in a region of highest values that is above the highest price the seller could ever want to offer. The seller cannot gain from setting prices above $\overline{v}(0)$, and thus she cannot gain from price dispersion.

Low-value externalities. We have assumed that the lowest buyer value is zero for all anticipated quantities; i.e., that F_q has support $[0, \overline{v}(q)]$ for all $q \in [0,1]$. Suppose instead that the support of F_q is $[\underline{v}(q), \overline{v}(q)]$, where $\underline{v}$ is continuously differentiable with $\underline{v}' \geq 0$. Adapting our concavity assumptions to this more general setting, we can show that our main results in Proposition 2 and Theorem 1 go through essentially unchanged. The proof of Theorem 1 would combine demand-preserving contractions as the ones we use in our baseline model together with some price increases below $\underline{v}(q^*)$, namely in a price range where all buyers are willing to purchase in equilibrium.

Concluding remarks. We have presented a framework for studying personalized pricing in markets with network externalities. Our analysis provides an explanation for the use of posted prices together with discounts that are dispersed across buyers. We showed how the seller's solution is shaped by her concern for strategic uncertainty, and how it changes with the strength of externalities and with heterogeneity across buyer groups.

We believe there are several potentially fruitful directions for future research. For example, one could build on our model to examine the possibility of congestion in consumption; this could be introduced by assuming that buyers' highest-value function $\overline{v}(q)$ is inverse-U-shaped in the anticipated quantity of trade q. Another interesting direction would be to extend our analysis to a two-sided platform, say with sellers on one side and buyers on the other.[32] Unlike in our heterogeneous-groups setting of Section 5.3, here participants on each side would have a value of participating that is increasing in the number

[32] See Jullien, Pavan and Rysman (2023) for a recent survey of the literature on two-sided markets with network effects.

of participants on the other side but (weakly) decreasing in the number of participants on their same side.

Finally, another possible direction would be to introduce dynamics. Suppose buyers receive offers from the seller and can hold onto them, so they decide not only whether to purchase but also when to purchase. Taking others' decisions to be independent of his own, assume a buyer purchases at a time $t \geq 0$ if and only if it is dominant for him to do so given the publicly observed quantity of purchases up until time $t-1$.[33] If the seller offers a constant price to each buyer, then her solution coincides with that in our static model. In fact, this dynamic setting offers a transparent dynamic implementation of our seller's solution that requires buyers to know neither the seller's price distribution nor the distribution of other buyers' types. In future work, we are interested in studying the conditions under which this solution remains optimal even when the seller can commit to prices that change over time.

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A. Preliminaries

Lemma 1. *Over* $\Delta(\mathbb{R}_+) \times [0,1]$*, endowing* $\Delta(\mathbb{R}_+)$ *with its weak* topology:*

- $R_q(\Pi)$ *is continuous in* (Π, q).
- $D_q(\Pi)$ *is continuous in* (Π, q) *where* $q > 0$*, and continuous in* q.

Proof. Fix $(\Pi, q) \in \Delta(\mathbb{R}_+) \times [0,1]$, at which we will establish continuity.

Consider first the case of $q = 0$. For demand, observe that any $p \in \mathbb{R}_+$ has $D_{\tilde{q}}(p) = D_0(p)$ for sufficiently small $\tilde{q} \in (0,1]$. Therefore, the Lebesgue dominated convergence theorem tells us $D_{\tilde{q}}(\Pi) \to D_0(\Pi)$ as $\tilde{q} \to 0$. For revenue, observe that any $(\tilde{\Pi}, \tilde{q}) \in \Delta(\mathbb{R}_+) \times [0,1]$ has $0 \leq R_{\tilde{q}}(\tilde{\Pi}) \leq \overline{v}(\tilde{q})$, so that $R_{\tilde{q}}(\tilde{\Pi}) \to 0 = R_0(\Pi)$ as $(\tilde{\Pi}, \tilde{q}) \to (\Pi, 0)$.

Now, we turn to the case of $q > 0$. Given a continuous and bounded function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, we will show $(\Pi, q) \mapsto \int \psi D_q \, \mathrm{d}\Pi$ is continuous on $\Delta(\mathbb{R}_+) \times (0,1]$. The demand and revenue results will follow by, respectively, setting $\psi(p) := 1$ and $\psi(p) := \min\{p,\ \overline{v}(1)\}$. Observe that any $(\tilde{\Pi}, \tilde{q}) \in \Delta(\mathbb{R}_+) \times (0,1]$ has

$$\begin{aligned} \left| \int \psi D_q \, \mathrm{d}\Pi - \int \psi D_{\tilde{q}} \, \mathrm{d}\tilde{\Pi} \right| &\leq \left| \int \psi D_q \, \mathrm{d}(\Pi - \tilde{\Pi}) \right| + \int \psi \left| D_q - D_{\tilde{q}} \right| \, \mathrm{d}\tilde{\Pi} \\ &\leq \left| \int \psi D_q \, \mathrm{d}(\Pi - \tilde{\Pi}) \right| + \|\psi\|_\infty \|D_q - D_{\tilde{q}}\|_\infty. \end{aligned}$$

Hence, because ψD_q is continuous and bounded on $\mathbb{R}_+$, we will have $\int \psi D_{\tilde{q}}\, \mathrm{d}\tilde{\Pi} \to \int \psi D_q\, \mathrm{d}\Pi$ as $\tilde{\Pi} \to \Pi$ and $\tilde{q} \to q$ if we establish the uniform distance $\|D_q - D_{\tilde{q}}\|_\infty$ converges to zero. To that end, define

$$\beta(\tilde{q}) := \Big| \|f_q\|_\infty - \|f_{\tilde{q}}\|_\infty \Big| \text{ and } \gamma(\tilde{q}) := \Big\| (f_q - f_{\tilde{q}})|_{[0,\ \min\{\overline{v}(q), \overline{v}(\tilde{q})\}]} \Big\|_\infty .$$

Now, every $p \in \mathbb{R}_+$ has

$$\begin{aligned} |D_q(p) - D_{\tilde{q}}(p)| &= \left| \left(\int_{[p,\infty)\cap[0,\overline{v}(q)]\cap[0,\overline{v}(\tilde{q})]} + \int_{[p,\infty)\cap \mathrm{co}\{\overline{v}(q),\overline{v}(\tilde{q})\}} \right) (f_q - f_{\tilde{q}}) \right| \\ &\leq \gamma(\tilde{q})\overline{v}(1) + |\overline{v}(q) - \overline{v}(\tilde{q})| \left[\|f_q\|_\infty + \beta(\tilde{q}) \right]. \end{aligned}$$

Because $\overline{v}$ is continuous and the right-hand side of the above inequality is independent of p, the lemma will follow if we establish that $\beta(\tilde{q})$ and $\gamma(\tilde{q})$ both converge to zero as $\tilde{q} \to q$. So given any $\varepsilon > 0$, we want to show $\tilde{q} \in (0,1]$ close enough to q has $\beta(\tilde{q}), \gamma(\tilde{q}) < \varepsilon$.

Let $Q := [\frac{1}{2}q, 1]$, a compact neighborhood of q in $(0,1]$. Because a continuous function on a compact space is uniformly continuous, some $\delta > 0$ exists such that any $q_1, q_2 \in Q$ and $v_1 \in [0, \overline{v}(q_1)]$, $v_2 \in [0, \overline{v}(q_2)]$ such that $|q_1 - q_2|, |v_1 - v_2| < \delta$ have $|f_{q_1}(v_1) - f_{q_2}(v_2)| < \frac{\varepsilon}{2}$. But then, consider any $\tilde{q} \in Q$ with $|\tilde{q} - q|, |\overline{v}(\tilde{q}) - \overline{v}(q)| < \delta$—close enough $\tilde{q}$ satisfies these inequalities because $\overline{v}$ is continuous. Clearly, $\gamma(\tilde{q}) \leq \frac{\varepsilon}{2} < \varepsilon$, so it remains to show $\beta(\tilde{q}) < \varepsilon$.

Take any $\{q_1, q_2\} = \{q, \tilde{q}\}$. Some $v_1 \in [0, \overline{v}(q_1)]$ exists such that $f_{q_1}(v_1) = \|f_{q_1}\|_\infty$. But then, because $v_2 := \min\{v_1, \overline{v}(q_2)\}$ has $|v_1 - v_2| < \delta$, we have

$$\|f_{q_1}\|_\infty = f_{q_1}(v_1) \leq |f_{q_1}(v_1) - f_{q_1}(v_2)| + |f_{q_1}(v_2) - f_{q_2}(v_2)| + f_{q_2}(v_2) < 2\tfrac{\varepsilon}{2} + \|f_{q_2}\|_\infty.$$

Hence, $\beta(\tilde{q}) < \varepsilon$, as required. *Q.E.D.*

Lemma 2. *For any given price distribution $\Pi \in \Delta(\mathbb{R}_+)$, the seller's revenue is weakly increasing in the anticipated quantity, strictly so wherever the revenue is strictly positive. Moreover, the induced set of equilibrium quantities is closed and nonempty. Hence, a least-quantity equilibrium exists and is the unique worst-case equilibrium.*

Proof. By Lemma 1, the set of equilibrium quantities is closed. Because $D_0(\Pi) \geq 0$ and $D_1(\Pi) \leq 1$, it follows from Lemma 1 and the intermediate value theorem that some equilibrium quantity exists.

Toward the payoff ranking, note an anticipated quantity q generates revenue

$$R_q(\Pi) = \int_0^\infty pD_q(p)\ \mathrm{d}\Pi(p).$$

Because the integrand weakly increases with $D_q(p)$ at every $p \geq 0$, it weakly increases (given monotonicity of u) with q.

Now we pursue the strict revenue ranking. Suppose two quantities $\tilde{q}, q \in [0,1]$ have $\tilde{q} < q$ and $R_q(\Pi) > 0$. We want to show $R_q(\Pi) > R_{\tilde{q}}(\Pi)$. The claim holds if $\tilde{q} = 0$ because then $R_{\tilde{q}}(\Pi) = 0$; so focus on the alternative case. In this case, we can pair Assumption 1 with the fact that (given $\overline{v}' > 0$) the distributions F_q and $F_{\tilde{q}}$ are not identical, to deduce $D_q(p) > D_{\tilde{q}}(p)$ for every $p \in (0, \overline{v}(q))$. That $R_q(\Pi) > 0$ implies Π puts positive mass on such prices then implies $\int_0^\infty pD_q(p)\ \mathrm{d}\Pi(p) > \int_0^\infty pD_{\tilde{q}}(p)\ \mathrm{d}\Pi(p)$, as desired.

Having shown the set of equilibrium quantities is closed in the compact set $[0,1]$, a lowest equilibrium quantity q exists. We also know q is a worst-case equilibrium quantity, uniquely so if $R_q(\Pi) > 0$. Finally, if q is a zero-revenue equilibrium quantity, then our tie-breaking assumption implies $q = \Pi(0)$, and so q is the unique zero-revenue equilibrium quantity. The lemma follows. *Q.E.D.*

B. Proofs for Section 3 and Section 4

B.1. Proof of Proposition 1

First, let us show that a best degenerate-price equilibrium exists and generates strictly positive revenue. To that end, consider the program

$$\begin{aligned} \max_{(p,q)\in[0,\overline{v}(1)]\times[0,1]} \quad & pq \\ \text{s.t.} \quad & q\left[D_q(p) - q\right] = 0. \end{aligned}$$

First, let us observe the program admits some optimal solution (p^B, q^B) with strictly positive value. Indeed, notice the constraint function $(p, q) \mapsto q\,[D_q(p) - q]$ is continuous wherever the quantity is strictly positive by Lemma 1, and it is continuous at zero quantity because $(p, q) \mapsto D_q(p)$ is bounded. Therefore, the program has continuous objective on a compact domain and so admits an optimal solution (p^B, q^B). Moreover, because $(D_q^{-1}(q), q)$ is feasible and yields strictly positive value in the program for $q \in (0, 1)$, it follows that $p^B q^B > 0$.

Let us now see (p^B, q^B) is a best degenerate-price equilibrium. First, because $q^B > 0$, we know $D_{q^B}(p^B) - q^B = 0$, so q^B is an equilibrium quantity for the degenerate price distribution on p^B. Next, any alternative degenerate-price equilibrium (p, q) would either have $p > \overline{v}(1)$ and hence generate zero revenue, or would be feasible in the above program and so generate a weakly lower revenue.

It remains to show any nondegenerate price distribution $\Pi \in \Delta(\mathbb{R}_+)$, with any equilibrium quantity q it generates, does strictly worse than some degenerate-price equilibrium. If q is zero (and so too is revenue), then we have nothing to show because we have already shown a degenerate-price equilibrium can yield strictly positive revenue. So focus on the case of $q \in (0, 1]$. In this case, some uniform price—specifically $p = D_q^{-1}(q) \in [0, \overline{v}(q)]$—exists for which q is an equilibrium quantity. Moreover, because $\varphi_{q,q}$ is strictly increasing (given Assumption 3), the degenerate price yields a strictly higher revenue. *Q.E.D.*

B.2. Proof of Proposition 2

Toward showing this program's solutions are exactly the optimal pairs (Π^*, q^*), let us invest in some terminology. Say a pair $(\Pi, q) \in \Delta(\mathbb{R}_+) \times [0, 1]$ is *worst-feasible* if q is a worst equilibrium for the seller given price distribution Π. Say a pair $(\Pi^*, q^*) \in \Delta(\mathbb{R}_+) \times [0, 1]$ is *limit-worst-feasible (LWF)* if it is a limit of a sequence of worst-feasible pairs. Finally, let $R^* := \sup_{(\Pi,q)\text{ worst-feasible}} R_q(\Pi)$ denote the seller's optimal value.

Let us make three preliminary observations. First, any convergent sequence

$(\Pi_n, q_n)_{n=1}^{\infty}$ of worst-feasible pairs has

$$\lim_{n\to\infty} R_{q_n}(\Pi_n) = R_{q_\infty}(\Pi_\infty), \text{ where } (\Pi_\infty, q_\infty) = \lim_{n\to\infty}(\Pi_n, q_n) \tag{8}$$

by Lemma 1. Second, observe that some LWF pair $(\hat{\Pi}, \hat{q})$ has $R_{\hat{q}}(\hat{\Pi}) = R^*$. Indeed, to find such a pair, take some sequence of worst-feasible pairs $(q_n, \Pi_n)_{n=1}^{\infty}$ with $\lim_{n\to\infty} R_{q_n}(\Pi_n) = R^*$, which exists by definition of R^*. Because all prices in $[\overline{v}(1), \infty)$ yield the same revenue (zero), we can assume without loss that $\Pi_n \in \Delta[0, \overline{v}(1)]$. Then, by compactness, we can (dropping to a subsequence) assume without loss that $(\Pi_n, q_n)_{n=1}^{\infty}$ converges to some $(\hat{\Pi}, \hat{q})$—which is then as desired by (8). Third, some worst-feasible pair (hence some LWF pair) generates strictly positive revenue. Indeed, given $p \in (0, \overline{v}_{1/2})$, Lemma 2 implies one can pair $\Pi = \frac{1}{2}\mathbf{1}_{[0,\infty)} + \frac{1}{2}\mathbf{1}_{[p,\infty)}$ with its (strictly positive) lowest equilibrium quantity.

Let us now establish, given $(\Pi^*, q^*) \in [0,1] \times \Delta(\mathbb{R}_+)$, a four-way equivalence:

(i) The pair (Π^*, q^*) is optimal, in the sense defined in the main text.

(ii) The pair (Π^*, q^*) is LWF and has $R_{q^*}(\Pi^*) = R^*$.

(iii) The pair (Π^*, q^*) solves the program $\max_{(\Pi,q) \text{ LWF}} R_q(\Pi)$.

(iv) The pair (Π^*, q^*) solves program (P*).

Because we have noted above that some LWF pair $(\hat{\Pi}, \hat{q})$ has $R_{\hat{q}}(\hat{\Pi}) = R^*$, and because we have noted that some LWF pair generates strictly positive revenue, proving this four-way equivalence will prove the proposition. We will prove that (i) $\iff$ (ii) $\iff$ (iii) $\iff$ (iv). First, note that (i) $\iff$ (ii) follows immediately from (8).

Now, let us see that (ii) $\iff$ (iii). Recall that some some LWF pair $(\hat{q}, \hat{\Pi})$ has $R_{\hat{q}}(\hat{\Pi}) = R^*$. This equivalence will therefore follow if every LWF (Π, q) has $R_q(\Pi) \leq R^*$. And indeed, taking some sequence $(\Pi_n, q_n)_{n=1}^{\infty}$ of worst-feasible pairs converging to it, every n has $R_{q_n}(\Pi_n) \leq R^*$ by definition of R^*—but then $R_q(\Pi) \leq R^*$ by (8).

Finally, toward showing (iii) $\iff$ (iv), note that the two programs have the same objective, but different constraint sets. We will first show that any

LWF (Π, q) satisfies the constraints of program (P*). Then, we will show that any (Π, q) satisfying the constraints of program (P*) is either LWF or admits an alternative LWF pair $(\tilde{\Pi}, \tilde{q})$ that generates strictly higher revenue. This will imply the equivalence.

Take any LWF (Π, q), as witnessed by $(\Pi_n, q_n)_n$. Toward showing (Π, q) satisfies the constraints of program (P*), suppose $\hat{q} \in (0, q)$. For sufficiently large n, we have $q_n > \hat{q}$; let us argue that $D_{\hat{q}}(\Pi_n) > \hat{q}$ for such n, which will then imply $D_{\hat{q}}(\Pi) \geq \hat{q}$ by Lemma 1. To see we cannot have $D_{\hat{q}}(\Pi_n) \leq \hat{q}$, observe that $D_0(\Pi_n) \geq 0$, so that the intermediate value theorem would (given Lemma 1) deliver an equilibrium quantity in $[0, \hat{q}]$ for price distribution Π_n, in contradiction to the definition of q_n.

Finally, consider any (Π, q) satisfying the constraints of program (P*). We want to show either that (Π, q) is LWF or that an alternative LWF pair $(\tilde{\Pi}, \tilde{q})$ is a LWF generating strictly higher revenue. Because we know some LWF pair generates strictly positive revenue, the conclusion follows immediately if $R_q(\Pi) = 0$; so focus on the case of $R_q(\Pi) > 0$ from now on. Now, for any $\varepsilon \in (0, 1)$, define the price distribution $\Pi_\varepsilon := (1 - \varepsilon)\Pi + \varepsilon \mathbf{1}_{[0,\infty)}$. Then, every quantity $\hat{q} \in (0, q)$ has $D_{\hat{q}}(\Pi_\varepsilon) = (1 - \varepsilon)D_{\hat{q}}(\Pi) + \varepsilon \geq (1 - \varepsilon)\hat{q} + \varepsilon > \hat{q}$, and $D_0(\Pi_\varepsilon) \geq \varepsilon > 0$. In particular, the worst equilibrium for price distribution Π_ε is at least q. Now, considering some sequence $(\varepsilon_n)_n$ from $(0, 1)$ converging to zero, the sequence $(\Pi_{\varepsilon_n})_n$ of price distributions converges to Π, and has the property that the worst equilibrium quantity q_n for each price distribution Π_{ε_n} has $q_n \geq q$. Dropping to a subsequence if necessary, we may without loss assume q_n converges to some $\tilde{q} \in [q, 1]$ as $n \to \infty$. By construction, the pair $(\Pi, \tilde{q})$ is a LWF. If $\tilde{q} = q$ then (Π, q) is a LWF, and if $\tilde{q} > q$ then the LWF $(\Pi, \tilde{q})$ generates strictly higher revenue than (Π, q) by Lemma 2. The proposition follows. *Q.E.D.*

B.3. Inputs for the proof of Theorem 1

The following lemma records a useful technical result that generalizes Proposition 4 of Rappoport (2024).

Lemma 3. *Suppose* $[\underline{v}, \overline{v}] \subset \mathbb{R}$ *is a nondegenerate interval;* $f, g : [\underline{v}, \overline{v}] \to \mathbb{R}$

are absolutely integrable functions; and $\psi : [\underline{v}, \overline{v}] \to \mathbb{R}$ *is a function of bounded variation.*[34]

(i) *Suppose* g *is zero wherever* f *is zero on* $[\underline{v}, \overline{v}]$*, and the ratio* $\frac{g}{f}$ *is weakly increasing where its denominator is nonzero. If* $\int_{\underline{v}}^{v} \psi f \geq 0$ *for every* $v \in [\underline{v}, \overline{v}]$*, with equality at* $v = \overline{v}$*, then* $\int_{\underline{v}}^{\overline{v}} \psi g \leq 0$*.*

(ii) *Suppose* g *is zero wherever* f *is zero on* $[\underline{v}, \overline{v}]$*, and the ratio* $\frac{g}{f}$ *is weakly increasing where its denominator is nonzero. If* $\int_{\underline{v}}^{v} \psi f \geq 0$ *for every* $v \in [\underline{v}, \overline{v}]$*, with equality at* $v = \overline{v}$*, and some* $v \in [\underline{v}, \overline{v}]$ *exists such that* $\int_{\underline{v}}^{v} \psi f > 0$ *and* $\frac{g}{f}$ *is not constant on any neighborhood of* v*, then* $\int_{\underline{v}}^{\overline{v}} \psi g < 0$*.*

(iii) *Suppose* f *is zero wherever* g *is zero on* $[\underline{v}, \overline{v}]$*, and the ratio* $\frac{f}{g}$ *is weakly decreasing where its denominator is nonzero. If* $f(\overline{v}), g(\overline{v}) \geq 0$ *and* $\int_{\underline{v}}^{v} \psi g \geq 0$ *for every* $v \in [\underline{v}, \overline{v}]$*, then* $\int_{\underline{v}}^{\overline{v}} \psi f \geq 0$*.*

Proof. First, we prove parts (i) and (ii). To that end, suppose the hypotheses of part (i) are satisfied. In what follows, we interpret $\frac{g}{f}$ as an arbitrary nondecreasing function $[\underline{v}, \overline{v}] \to \mathbb{R}$ that agrees with $\frac{g}{f}$ wherever f is nonzero. Now, define the absolutely continuous function $\Psi : [\underline{v}, \overline{v}] \to \mathbb{R}$ by letting $\Psi(v) := \int_{\underline{v}}^{v} \psi f$. Then, we can apply integration by parts for Stieltjes integration:

$$\begin{aligned}
\int_{\underline{v}}^{\overline{v}} \psi g &= \int_{\underline{v}}^{\overline{v}} \tfrac{g}{f} \Psi' \\
&= \left[\Psi \tfrac{g}{f}\right]_{\underline{v}}^{\overline{v}} - \int_{\underline{v}}^{\overline{v}} \Psi \, \mathrm{d}\tfrac{g}{f} \text{ (by integration by parts)} \\
&= 0 - \int_{\underline{v}}^{\overline{v}} \Psi \, \mathrm{d}\tfrac{g}{f} \text{ (since } \Psi(\underline{v}) = \Psi(\overline{v}) = 0) \\
&\leq 0 \text{ (since } \Psi \geq 0 \text{ and } \tfrac{g}{f} \text{ is weakly increasing)},
\end{aligned}$$

establishing part (i). Now, suppose in addition that some $v \in [\underline{v}, \overline{v}]$ exists such that $\Psi(v) > 0$ and $\frac{g}{f}$ is not constant on any neighborhood of v. By continuity,

[34] Throughout, for any interval $[\underline{v}, \overline{v}] \subseteq \mathbb{R}$ and any Lebesgue integrable function $h : [\underline{v}, \overline{v}] \to \mathbb{R}$, we let $\int_{\underline{v}}^{\overline{v}} h$ denote the Lebesgue integral $\int_{\underline{v}}^{\overline{v}} h(v) \, \mathrm{d}v$.

Ψ is strictly positive on some some nondegenerate interval of v. Because $\frac{g}{f}$ is not constant on this interval, it follows that $\int_{\underline{v}}^{\overline{v}} \psi g = -\int_{\underline{v}}^{\overline{v}} \Psi \ \mathrm{d}\frac{g}{f} < 0$, delivering (ii).

Next, we prove part (iii); suppose its hypotheses are satisfied. In what follows, we interpret $\frac{f}{g}$ as an arbitrary nonincreasing function $[\underline{v}, \overline{v}] \to \mathbb{R}$ that agrees with $\frac{f}{g}$ wherever g is nonzero.

Now, define the absolutely continuous functions $\Phi : [\underline{v}, \overline{v}] \to \mathbb{R}$ by letting $\Phi(v) := \int_{\underline{v}}^{v} \psi g$. Then,

$$\begin{aligned}
\int_{\underline{v}}^{\overline{v}} \psi f &= \int_{\underline{v}}^{\overline{v}} \tfrac{f}{g}\Phi' \\
&= \left[\Phi \tfrac{f}{g}\right]_{\underline{v}}^{\overline{v}} - \int_{\underline{v}}^{\overline{v}} \Phi \ \mathrm{d}\tfrac{f}{g} \text{ (by integration by parts)} \\
&= \Phi(\overline{v})\tfrac{f(\overline{v})}{g(\overline{v})} - \int_{\underline{v}}^{\overline{v}} \Phi \ \mathrm{d}\tfrac{f}{g} \text{ (since } \Phi(\underline{v}) = 0) \\
&\geq \Phi(\overline{v})\tfrac{f(\overline{v})}{g(\overline{v})} \text{ (since } \Phi \geq 0 \text{ and } \tfrac{f}{g} \text{ is weakly decreasing)} \\
&\geq 0 \text{ (since } \Phi(\overline{v}), f(\overline{v}), g(\overline{v}) \geq 0),
\end{aligned}$$

as required. *Q.E.D.*

The following lemma is a comparative statics result for comparing different price distributions: if a reduction in price dispersion preserves aggregate demand under a low anticipated quantity (and the only modified prices are those that will sometimes be exercised), then the reduction increases both demand and revenue when the anticipated quantity is higher.

Lemma 4. *Given* $q \in (0, 1]$, *suppose distinct price distributions* $\Pi, \tilde{\Pi} \in \Delta(\mathbb{R}_+)$ *are such that* $\Pi|_{(\overline{v}(q),\infty)} = \tilde{\Pi}|_{(\overline{v}(q),\infty)}$, *and*

$$\int_0^v (\Pi - \tilde{\Pi}) f_q \geq 0$$

for every $v \in [0, \overline{v}(q)]$, *with equality at* $v = \overline{v}(q)$.[35] *Then, any* $\tilde{q} \in [q, 1]$ *has*

$$D_{\tilde{q}}(\tilde{\Pi}) \geq D_{\tilde{q}}(\Pi) \text{ and } R_{\tilde{q}}(\tilde{\Pi}) > R_{\tilde{q}}(\Pi).$$

Proof. Both rankings can be derived as applications of Lemma 3, with $(\underline{v}, \overline{v}, f, \psi) = (0, \overline{v}_q, f_q, \Pi - \tilde{\Pi})$ and different choices of g.

First, consider $g := f_{\tilde{q}}|_{[0, \overline{v}(q)]}$, and apply Assumption 1. By Lemma 3(i),[36]

$$\begin{aligned} 0 &\leq \int_0^{\overline{v}(q)} (\tilde{\Pi} - \Pi) f_{\tilde{q}} \\ &= \int_0^{\overline{v}(\tilde{q})} (\tilde{\Pi} - \Pi) f_{\tilde{q}} \\ &= D_{\tilde{q}}(\tilde{\Pi}) - D_{\tilde{q}}(\Pi). \end{aligned}$$

Next, consider $g := \varphi_{q,\tilde{q}} f_q$. As Assumption 3 holds, Lemma 3(ii) tells us

$$\begin{aligned} 0 &< \int_0^{\overline{v}(q)} (\tilde{\Pi} - \Pi) \varphi_{q,\tilde{q}} f_q \\ &= \int_0^{\overline{v}(\tilde{q})} (\tilde{\Pi} - \Pi) \varphi_{\tilde{q},\tilde{q}} \, \mathrm{d}F_{\tilde{q}} \\ &= \int_0^{\overline{v}(\tilde{q})} (\Pi - \tilde{\Pi}) \, \mathrm{d}R_{\tilde{q}} \\ &= 0 - \int R_{\tilde{q}} \, \mathrm{d}(\Pi - \tilde{\Pi}) \\ &= R_{\tilde{q}}(\tilde{\Pi}) - R_{\tilde{q}}(\Pi). \end{aligned}$$

Q.E.D.

The following lemma extends our concave externalities assumption to price distributions rather than just prices.

Lemma 5. *Suppose* $\Pi \in \Delta(\mathbb{R}_+)$ *and* $0 \leq q_0 < q_1 \leq 1$ *have* $\Pi(\overline{v}(q_0)) = \Pi(\overline{v}(q_1)^-)$. *Then* $q \mapsto D_q(\Pi)$ *is concave on* $[q_0, q_1]$, *strictly so if* $\Pi(\overline{v}(q_0)) > 0$.

[35] Note, the equality at $\overline{v}(q)$ says exactly that $D_q(\tilde{\Pi}) = D_q(\Pi)$.

[36] One can alternatively prove this ranking by using the fact that $F_{\tilde{q}} \circ F_q^{-1}$ is convex under Assumption 1.

Proof. For any price $p \leq \overline{v}(q_0)$, the function $q \mapsto D_q(p)$ is strictly concave on (q_0, q_1) by Assumption 2, hence on $[q_0, q_1]$ by Lemma 1. For any price $p \geq \overline{v}(q_1)$, the function $q \mapsto D_q(p)$ is zero on $[q_0, q_1]$. Because a pointwise weighted average of concave functions is concave, strictly so if this average puts strictly positive weight on strictly concave functions, the lemma follows. *Q.E.D.*

To state the next lemma, we invest in some notation.

Notation 1.

- *Let* $\dot{f}_q(v)$ *denote the partial derivative of* $f_q(v)$ *with respect to* q*, which exists wherever* $q \in (0,1]$ *and* $0 \leq v \leq \overline{v}(q)$*.*
- *Let* $\partial D_q(\Pi)$ *[resp.* $\partial^- D_q(\Pi)$ *or* $\partial^+ D_q(\Pi)$*] denote the partial derivative [resp. left derivative or right derivative] of* $D_q(\Pi)$ *with respect to* q*, if it exists.*

The following lemma establishes that one-sided derivatives of demand with respect to anticipated quantity are finite, and that the demand function is kinked if and only if the price distribution has a mass point.

Lemma 6. *Suppose* $\Gamma : [0, \overline{v}(1)) \to \mathbb{R}_+$ *is increasing and right continuous, and* $q \in (0,1]$*. Then:*

- $\partial^- D_q(\Gamma) = \int_0^{\overline{v}(q)} \Gamma \dot{f}_q + \overline{v}'(q)\Gamma\left(\overline{v}(q)^-\right) f_q\left(\overline{v}(q)\right) \in \mathbb{R}$.
- *If* $q < 1$*, then* $\partial^+ D_q(\Pi) = \int_0^{\overline{v}(q)} \Gamma \dot{f}_q + \overline{v}'(q)\Gamma\left(\overline{v}(q)\right) f_q\left(\overline{v}(q)\right) \in \mathbb{R}$.
- *If* Γ *is continuous at* $\overline{v}(q)$*, then* $\tilde{q} \mapsto D_{\tilde{q}}(\Gamma)$ *is differentiable at* q*.*
- *If* $q < 1$ *and* Γ *is discontinuous at* $\overline{v}(q)$*, then* $\tilde{q} \mapsto D_{\tilde{q}}(\Gamma)$ *has a convex kink at* q*.*

Proof. Whenever $0 \leq q_0 < q_1 \leq 1$, we have

$$\begin{aligned}
\tfrac{D_{q_1}(\Gamma) - D_{q_0}(\Gamma)}{q_1 - q_0} &= \tfrac{1}{q_1 - q_0}\left[\int_0^{\overline{v}(q_1)} \Gamma f_{q_1} - \int_0^{\overline{v}(q_0)} \Gamma f_{q_0}\right] \\
&= \int_0^{\overline{v}(q_0)} \Gamma \tfrac{f_{q_1} - f_{q_0}}{q_1 - q_0} + \tfrac{\overline{v}(q_1) - \overline{v}(q_0)}{q_1 - q_0} \tfrac{1}{\overline{v}(q_1) - \overline{v}(q_0)} \int_{\overline{v}(q_0)}^{\overline{v}(q_1)} \Gamma f_{q_1}.
\end{aligned}$$

Given the Lebesgue dominated convergence theorem, the first two points come from applying this expression as $q_0 \nearrow q = q_1$ and as $q_1 \searrow q = q_0$, respectively. Then, combine the first two points for $q \in (0, 1)$ to obtain

$$\partial^+ D_q(\Gamma) - \partial^- D_q(\Gamma) = \overline{v}'(q) f_q(\overline{v}(q)) \left[\Gamma\left(\overline{v}(q)\right) - \Gamma\left(\overline{v}(q)^-\right)\right],$$

directly implying the last two points. *Q.E.D.*

B.4. Proof of Theorem 1

We begin with some useful terminology.

Definition 4. *Consider any price distribution* Π*. Given* $q \in [0, 1]$*:*

- *Say* Π ***has mass at*** q^{++} *if* $\Pi(p) > \Pi\left(\overline{v}(q)\right)$ *for every* $p > \overline{v}(q)$*.*
- *Say* Π ***has mass at*** q^{--} *if* $\Pi(p) < \Pi\left(\overline{v}(q)^-\right)$ *for every* $p < \overline{v}(q)$*.*
- *Say* Π ***has mass at*** q^+ *[resp.* q^-*] if it has a mass at* q^{++} *[resp. at* q^{--}*] or has a mass point at* $\overline{v}(q)$*.*

Given $q_0, q_1 \in [0, 1]$ *with* $q_0 < q_1$*, say* Π *is* ***degenerate on*** $[q_0, q_1]$ *if some* $p \in [\overline{v}(q_0), \overline{v}(q_1)]$ *exists such that* $\Pi\left(p^-\right) = \Pi\left(\overline{v}(q_0)^-\right)$ *and* $\Pi\left(p\right) = \Pi\left(\overline{v}(q_1)\right)$*.*

The following claim shows any optimal price distribution in the subproblem associated with any targeted quantity uses only prices below the monopoly price for that anticipated quantity's demand curve.

Claim 1. *Suppose* $\Pi \in \Delta(\mathbb{R}_+)$ *and* $\hat{q} \in (0, 1]$ *have* $\Pi\left(p^M(\hat{q})\right) < 1$*. Then, some* $\tilde{\Pi} \in \Delta(\mathbb{R}_+)$ *exists such that* $D_q(\tilde{\Pi}) \geq D_q(\Pi)$ *for every* $q \in [0, 1]$*, and* $R_{\hat{q}}(\tilde{\Pi}) > R_{\hat{q}}(\Pi)$*.*

Proof. Let $p^* := p^M(\hat{q})$, and let $\tilde{\Pi} := \Pi|_{[0,p^*)} \cup \mathbf{1}|_{[p^*,\infty)}$. The distribution $\tilde{\Pi}$ is below Π in the sense of first-order stochastic dominance, so that $D_q(\tilde{\Pi}) \geq D_q(\Pi)$ for every $q \in [0, 1]$. Moreover, Assumption 3 implies any price $p \neq p^*$ has $R_{\hat{q}}(p) < R_{\hat{q}}(p^*)$. Therefore, given that $\Pi(p^*) < 1$, we have

$$R_{\hat{q}}(\tilde{\Pi}) - R_{\hat{q}}(\Pi) = \int_{p^*}^{\infty} \left[R(p^*) - R(p)\right] \, \mathrm{d}\Pi(p) > 0,$$

as desired. *Q.E.D.*

The following claim uses concave externalities to establish that the slack on the demand constraint is first-order wherever the price distribution has a gap at the edge of a slack region.

Claim 2. *Suppose* $\Pi \in \Delta(\mathbb{R}_+)$ *and* $q \in [0,1]$ *have* $D_q(\Pi) = q$*.*

- *If* $q < 1$*, every* $\tilde{q} > q$ *close enough to* q *has* $D_{\tilde{q}}(\Pi) > \tilde{q}$*, and* Π *has no mass at* q^{++}*, then* $\partial^+ D_q(\Pi) > 1$*.*
- *If* $q > 0$*, every* $\tilde{q} < q$ *close enough to* q *has* $D_{\tilde{q}}(\Pi) > \tilde{q}$*, and* Π *has no mass at* q^{--}*, then* $\partial^- D_q(\Pi) < 1$*.*

Proof. Define the function $\psi : [0,1] \to \mathbb{R}$ via $\psi(\tilde{q}) := D_{\tilde{q}}(\Pi) - \tilde{q}$, which is continuous by Lemma 1. By Lemma 5, we know ψ is concave in an interval to the right [left] of q if $q < 1$ [resp. $q > 0$] and Π has no mass at q^{++} [resp. q^{--}].

Now, if ψ is zero at q and concave and strictly positive in a right [resp. left] neighborhood of q, it follows that its right [resp. left] derivative at q is strictly positive [resp. strictly negative], delivering the claim. *Q.E.D.*

The following claim shows that a feasible price distribution is always nondegenerate over (the closure of) any slack region in the range of its support.

Claim 3. *Suppose* $\Pi \in \Delta(\mathbb{R}_+)$ *and* $p^* := \max \operatorname{supp} \Pi$ *has* $D_q(\Pi) \geq q$ *for every* $q \in (0, \underline{q}(p^*))$*. If* (q_0, q_1) *is a connected component of*

$$\left\{ q \in \big(0, \underline{q}(p^*)\big) : \; D_q(\Pi) > q \right\},$$

then Π *is nondegenerate on* $[q_0, q_1]$*.*

Proof. The claim holds vacuously if $p^* = 0$, so focus on the case in which $p^* > 0$.

If Π has mass at q_0^{++} or at q_1^{--}, it is clearly nondegenerate on $[q_0, q_1]$. So now, focus on the case in which Π has mass neither at q_0^{++} nor at q_1^{--}. The claim will now follow if we establish that Π has mass points both at $\overline{v}(q_0)$ and at $\overline{v}(q_1)$.

Observe first that $\min\operatorname{supp}\Pi = 0$, for otherwise small enough $q \in \big(0, \underline{q}(p^*)\big)$ will have $D_q(\Pi) = 0 < q$. Then, by definition of the support (and the hypothesis that Π has mass neither at q_0^{++} nor at q_1^{--}), we know that Π has a mass point at $0 = \overline{v}(q_0)$ if $q_0 = 0$, and has a mass point at $p^* = \overline{v}(q_1)$ if $q_1 = \underline{q}(p^*)$.

It remains now to show that Π has a mass point at $\overline{v}(q_0)$ if $q_0 > 0$, and has a mass point at $\overline{v}(q_1)$ if $q_1 < \underline{q}(p^*)$. So suppose $q_0 > 0$ [resp. $q_1 < \underline{q}(p^*)$]. By definition of (q_0, q_1), no $\tilde{q}_0 < q_0$ [resp. $\tilde{q}_1 > q_1$] exists such that every $q \in [\tilde{q}_0, q_0]$ [resp. every $q \in [q_1, \tilde{q}_1]$] has $D_q(\Pi) > q$. But then, by Lemma 1 we in fact have that $D_{q_0}(\Pi) = q_0$ [resp. $D_{q_1}(\Pi) = q_1$]. Claim 2 thus implies $\partial^+ D_{q_0}(\Pi) > 1$ [resp. $\partial^- D_{q_1}(\Pi) < 1$]. Meanwhile, that $q \mapsto D_q(\Pi) - q$ is zero at q_0 [resp. q_1] and nonnegative just to the left [resp. right] of it implies $\partial^- D_{q_0}(\Pi) \leq 1$ [resp. $\partial^+ D_{q_1}(\Pi) \geq 1$]. Thus, $q \mapsto D_q(\Pi)$ has a convex kink at q_0 [resp. q_1], and so Lemma 6 tells us Π has a mass point at $\overline{v}(q_0)$ [resp. $\overline{v}(q_1)$] as desired. *Q.E.D.*

The following claim says that whenever the price distribution is nondegenerate over some interval, a smaller such interval can be found on which the price distribution is also well-behaved.

Claim 4. *Suppose $\Pi \in \Delta(\mathbb{R}_+)$ and $0 \leq q_0 < q_1 \leq 1$ are such that Π is nondegenerate on $[q_0, q_1]$. Then some $\tilde{q}_0, \tilde{q}_1 \in [q_0, q_1]$ with $\tilde{q}_0 < \tilde{q}_1$ exist such that:*

- Π *is nondegenerate on* $[\tilde{q}_0, \tilde{q}_1]$*;*
- *either* $\tilde{q}_0 \in (q_0, q_1)$ *or* Π *has no mass at* q_0^{++}*;*
- *either* $\tilde{q}_1 \in (q_0, q_1)$ *or* Π *has no mass at* q_1^{--}*.*

Proof. If Π has mass at q_1^{--}, then any $\tilde{q}_0 \in (q_0, q_1)$, paired with any $\tilde{q}_1 \in (\tilde{q}_0, q_1)$ close enough to q_1, is as desired. If Π has mass at q_0^{++}, then any $\tilde{q}_1 \in (q_0, q_1)$, paired with any $\tilde{q}_0 \in (q_0, \tilde{q}_1)$ close enough to q_0, is as desired. If Π has no mass at q_1^{--} or at q_0^{++}, then $\tilde{q}_0 = q_0$ and $\tilde{q}_1 = q_1$ are as desired. *Q.E.D.*

The following claim shows that small enough perturbations preserve the demand constraint on any interval where it is slack (with first-order slack at the edges).

Claim 5. *Suppose $\Pi, \tilde{\Pi} \in \Delta(\mathbb{R}_+)$ and $0 \leq q_0 < q_1 \leq 1$ are such that:*

- *Every* $q \in (q_0, q_1)$ *has* $D_q(\Pi) > q$, *and each* $q \in \{q_0, q_1\}$ *has* $D_q(\tilde{\Pi}) \geq q$;
- *Either* $D_{q_0}(\Pi) > q_0$ *or* $\partial^+ D_{q_0}(\Pi) > 1$, *with the former case if* $q_0 = 0$; *and*
- *Either* $D_{q_1}(\Pi) > q_1$ *or* $\partial^- D_{q_1}(\Pi) < 1$.

Then, letting $\Pi_\varepsilon := (1-\varepsilon)\Pi + \varepsilon\tilde{\Pi}$, *any small enough* $\varepsilon \in (0,1)$ *has*

$$D_q(\Pi_\varepsilon) \geq q, \ \forall q \in [q_0, q_1].$$

Proof. First, for either $q \in \{q_0, q_1\}$, if $D_q(\Pi) > q$, then (given that $D_q(\tilde{\Pi}) \geq q$) every $\varepsilon \in (0,1)$ has $D_q(\Pi_\varepsilon) > q$. Next, if either $q \in \{q_0, q_1\}$ has $D_q(\Pi) = q$ (which in particular means $q > 0$ given our hypotheses), then Lemma 6 tells us one-sided derivatives of $\tilde{q} \mapsto D_{\tilde{q}}(\tilde{\Pi})$ are finite there. So, for small enough $\overline{\varepsilon} \in (0,1)$:

- Each $q \in \{q_0, q_1\}$ has $D_q(\Pi_{\overline{\varepsilon}}) \geq q$;
- Either $D_{q_0}(\Pi_{\overline{\varepsilon}}) > q_0$ or $\partial^+ D_{q_0}(\Pi_{\overline{\varepsilon}}) > 1$; and
- Either $D_{q_1}(\Pi_{\overline{\varepsilon}}) > q_1$ or $\partial^- D_{q_1}(\Pi_{\overline{\varepsilon}}) < 1$.

Fixing such an $\overline{\varepsilon}$, some $\tilde{q}_0, \tilde{q}_1 \in (q_0, q_1)$ exist such that every $q \in (q_0, \tilde{q}_0] \cup [\tilde{q}_1, q_1)$ has $D_q(\Pi_{\overline{\varepsilon}}) > q$. Hence, because $\varepsilon \mapsto D_q(\Pi_\varepsilon)$ is affine for every q, it follows that every $q \in (q_0, \tilde{q}_0] \cup [\tilde{q}_1, q_1)$ and $\varepsilon \in (0, \overline{\varepsilon}]$ have $D_q(\Pi_{\overline{\varepsilon}}) \geq q$.

Hence, all that remains is to see (focusing on the nontrivial case that $q_0 < q_1$) that sufficiently small $\varepsilon \in (0, \overline{\varepsilon}]$ have $D_q(\Pi_\varepsilon) \geq q$ for every $(\tilde{q}_0, \tilde{q}_1)$. And indeed, given Lemma 1, Berge's theorem tells us the function $[0, \overline{\varepsilon}] \to \mathbb{R}$ given by $\varepsilon \mapsto \min_{q \in [\tilde{q}_0, \tilde{q}_1]} [D_q(\Pi_\varepsilon)]$ is well-defined and continuous. Because $[\tilde{q}_0, \tilde{q}_1] \subset (q_0, q_1)$, this function is strictly positive at $\varepsilon = 0$, and so is strictly positive for small enough $\varepsilon \in (0, \overline{\varepsilon}]$, delivering the claim. *Q.E.D.*

Now, with these claims in hand, we pursue the proof of the theorem.

Proof of Theorem 1. First, given Claim 1, any optimal (Π^*, q^*) must have $\max \operatorname{supp} \Pi^* \leq p^M(q^*)$.

Now, we show that any optimal (Π^*, q^*) has Π^* greedy up to the top of its support. To that end, consider $q^* \in [0,1]$ and $\Pi \in \Delta(\mathbb{R}_+)$ such that $D_q(\Pi) \geq q$

for every $q \in (0, q^*)$, and Π is not greedy up to $p^* := \max \operatorname{supp} \Pi$. We want to show (Π, q^*) cannot be optimal. We have nothing to show (given the previous paragraph) if $p^* \geq \overline{v}(q^*)$, so without loss say $p^* < \overline{v}(q^*)$. Now, by hypothesis, the set

$$\left\{q \in \left(0, \underline{q}(p^*)\right) : \ D_q(\Pi) > q\right\}$$

is nonempty. Meanwhile, Lemma 1 implies this set is open in $\mathbb{R}$, and so every connected component of it is an open interval. Let (q_0, q_1) be such a connected component. Claim 3 (which applies because $p^* \leq \overline{v}(q^*)$) tells us Π is nondegenerate on $[q_0, q_1]$. Hence, Claim 4 delivers some $\tilde{q}_0, \tilde{q}_1 \in [q_0, q_1]$ with $\tilde{q}_0 < \tilde{q}_1$ such that Π is nondegenerate on $[\tilde{q}_0, \tilde{q}_1]$; either $\tilde{q}_0 \in (q_0, q_1)$ or Π has no mass at q_0^{++}; and either $\tilde{q}_1 \in (q_0, q_1)$ or Π has no mass at q_1^{--}. Moreover, by Lemma 1, we know $D_{q_0}(\Pi) \geq q_0$ and $D_{q_1}(\Pi) \geq q_1$. Hence, applying Claim 2, we therefore have that either $D_{\tilde{q}_0}(\Pi) > \tilde{q}_0$ or $\partial^+ D_{\tilde{q}_0}(\Pi) > 1$; and either $D_{\tilde{q}_1}(\Pi) > \tilde{q}_1$ or $\partial^- D_{\tilde{q}_1}(\Pi) < 1$. So given any $\tilde{\Pi} \in \Delta(\mathbb{R}_+)$ with $D_{\tilde{q}_0}(\tilde{\Pi}) \geq \tilde{q}_0$ and $D_{\tilde{q}_1}(\tilde{\Pi}) \geq \tilde{q}_1$, Claim 5 tells us sufficiently small $\varepsilon \in (0, 1)$ has $D_q\left((1-\varepsilon)\Pi + \varepsilon\tilde{\Pi}\right) \geq q$ for every $q \in [\tilde{q}_0, \tilde{q}_1]$.

We are now equipped to show (Π, q^*) is suboptimal. For any $p \in [\overline{v}(\tilde{q}_0), \overline{v}(\tilde{q}_1)]$, consider the price distribution Π^p which coincides with Π on $[0, \overline{v}(\tilde{q}_0)) \cup [\overline{v}(\tilde{q}_1), \infty)$, takes value $\Pi\left(\overline{v}(\tilde{q}_0)^-\right)$ on $[\overline{v}(\tilde{q}_0), p)$, and takes value $\Pi\left(\overline{v}(\tilde{q}_1)\right)$ on $[p, \overline{v}(\tilde{q}_1))$. Observe that $p \mapsto D_{q_1}(p)$ is decreasing, Π lies between $\Pi^{\overline{v}(\tilde{q}_0)}$ and $\Pi^{\overline{v}(\tilde{q}_1)}$ (in the sense of first-order stochastic dominance), and $p \mapsto D_{q_1}(\Pi^p)$ is continuous by Lemma 1. Hence, the intermediate value function yields some $p \in [\overline{v}(\tilde{q}_0), \overline{v}(\tilde{q}_1)]$ such that $D_{q_1}(\Pi^p) = D_{q_1}(\Pi)$. For any $\varepsilon \in (0, 1)$, let $\Pi_\varepsilon := (1-\varepsilon)\Pi + \varepsilon\Pi^p$. By construction, every $q \in (0, q_0]$ has $D_q(\Pi^p) = D_q(\Pi) \geq q$. Meanwhile Lemma 4 tells us $R_{q^*}(\Pi^p) > R_{q^*}(\Pi)$ and every $q \in [q_1, q^*)$ has $D_q(\Pi^p) \geq D_q(\Pi) \geq q$. Therefore, for any $\varepsilon \in (0, 1)$, we have $R_{q^*}(\Pi_\varepsilon) > R_{q^*}(\Pi)$ and $D_q(\Pi^p) \geq q$ for every $q \in (0, q_0] \cup [q_1, q^*)$. Finally, as noted in the previous paragraph, sufficiently small $\varepsilon \in (0, 1)$ has $D_q(\Pi^p) \geq q$ for every $q \in [\tilde{q}_0, \tilde{q}_1]$. So (Π^ε, q^*) witnesses that (Π, q^*) is suboptimal, as claimed above.

Now, letting (Π^*, q^*) be optimal and $p^* := \max \operatorname{supp} \Pi^*$, we have established that Π^* is greedy up to p^* and $p^* \leq p^M(q^*)$. All that remains is to see Π^* has a mass point at p^*. To that end, note that $p^* \leq p^M(q^*) < \overline{v}(q^*)$ implies

$\underline{q}(p^*) < q^*$. We can therefore apply Claim 2 to learn $\partial^+ D_{\underline{q}(p^*)}(\Pi^*) > 1$. But greediness up to p^* directly tells us $\partial^- D_{\underline{q}(p^*)}(\Pi^*) = 1$, and so Lemma 6 implies Π^* has a mass point at $\underline{q}(p^*)$. *Q.E.D.*

B.5. Proof of Corollary 1

Corollary 1 follows directly from Theorem 1 and the next Lemma 7, which shows that greediness rules out mass points and gaps in a price distribution.

Lemma 7. *Suppose* $\hat{q} \in (0, 1]$ *and* $\Gamma : [0, \overline{v}(\hat{q})) \to \mathbb{R}_+$ *is increasing and right continuous with* $D_q(\Gamma) = q$ *for every* $q \in (0, \hat{q})$. *Then* Γ *is strictly increasing and continuous on* $[0, \overline{v}(\hat{q}))$ *with* $\Gamma(0) = 0$.

Proof. By hypothesis, $q \mapsto D_q(\Gamma)$ is differentiable on $(0, \hat{q})$, and so Lemma 6 tells us Γ has no discontinuities in $(0, \overline{v}(\hat{q}))$. Moreover, $\Gamma(0) = D_0(\Gamma) = 0$.

To show Γ is strictly increasing on $[0, \overline{v}(\hat{q}))$, it suffices to show (given that it is weakly increasing by definition) that it is not constant over any interval. So suppose $0 < q_0 < q_1 < \overline{v}(\hat{q})$. Because $\Gamma(q_0) = q_0 > 0$, Lemma 5 would imply $q \mapsto D_q(\Gamma)$ is strictly concave if $\Gamma \circ \overline{v}$ were constant on (q_0, q_1). But this function is linear by hypothesis, hence not strictly concave. It follows that $\Gamma \circ \overline{v}$ is constant on (q_0, q_1), delivering the claim. *Q.E.D.*

Supplementary Appendix

C. Proofs for Section 5

C.1. Preliminaries

Lemma 8. *In the linear demand environment, define* $\Gamma^* : [0, \overline{v}(1)] \to \mathbb{R}$ *by*

$$\Gamma^*(v) := \underline{q}(v) + \frac{v}{\overline{v}'(\underline{q}(v))}.$$

(i) The function Γ^* *is continuous and strictly increasing, and every* $q \in [0,1]$ *has*

$$\overline{v}(q) D_q(\Gamma^*) = \int_0^{\overline{v}(q)} \Gamma^* = q\overline{v}(q).$$

(ii) The function Γ^* *is greedy.*[37] *Conversely, if* $\hat{q} \in [0,1]$ *and* Γ *is greedy up to* $\overline{v}(\hat{q})$*, then* Γ *agrees with* Γ^* *on* $[0, \overline{v}(\hat{q}))$*.*

(iii) A unique $\overline{p}^* \in (0, \overline{v}(1))$ *exists with* $\Gamma^*(\overline{p}^*) = 1$*.*

(iv) Every $\hat{p} \in [0, \overline{p}^*]$ *admits a unique* $\hat{q} \in [\underline{q}(\overline{p}^*), 1]$ *such that* $\int_{\hat{p}}^{\overline{v}(\hat{q})} (1 - \Gamma^*) = 0$*, and this* $\hat{q}$ *strictly decreases as* $\hat{p}$ *increases.*

Proof. First, let us observe that Γ^* is continuous and weakly increasing. It is continuous because $\overline{v}$ is continuously differentiable and $\overline{v}'$ is strictly positive. To see it is weakly increasing (or equivalently, that $\Gamma^* \circ \overline{v}$ is) we apply convexity of $\overline{v}$. First, consider the case in which $\overline{v}$ is twice differentiable. In this case, on $(0,1]$ (where $\overline{v}, \overline{v}'$ are both strictly positive), we have[38]

$$\begin{aligned}
(\Gamma \circ \overline{v})' &= 1 + \left(\tfrac{\overline{v}}{\overline{v}'}\right)' = 1 + \tfrac{(\overline{v}')^2 - \overline{v}\ \overline{v}''}{(\overline{v}')^2} = \tfrac{2(\overline{v}')^2 - \overline{v}\ \overline{v}''}{(\overline{v}')^2} \\
&= \tfrac{\overline{v}^3}{(\overline{v}')^2} \tfrac{2(\overline{v}')^2 - \overline{v}\ \overline{v}''}{\overline{v}^3} = \tfrac{\overline{v}^3}{(\overline{v}')^2} \tfrac{2\overline{v}\ \overline{v}'\overline{v}' - \overline{v}^2 \overline{v}''}{\overline{v}^4} = \tfrac{\overline{v}^3}{(\overline{v}')^2} \left(\tfrac{-\overline{v}'}{\overline{v}^2}\right)' = \tfrac{\overline{v}^3}{(\overline{v}')^2} \left(\tfrac{1}{\overline{v}}\right)'' \\
&\geq 0,
\end{aligned}$$

[37] Our main text defines greediness only for (increasing and right-continuous) functions $\mathbb{R}_+ \to \mathbb{R}_+$, but the definition can be applied verbatim to a function defined on $[0, \overline{v}(1)]$.

[38] This argument is substantively the same as the observation (McAfee and McMillan, 1987, footnote 11) that a type distribution is regular if and only if the inverse of its survival function is convex.

where the last inequality holds because $\frac{1}{\overline{v}}$ is convex. The general case—in which $\overline{v} : [0,1] \to \mathbb{R}$ is an arbitrary continuously differentiable function that is zero at zero, has strictly positive derivative, and has $\frac{1}{\overline{v}}$ convex on $(0,1]$—follows from an approximation argument.[39]

Now, let $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$ be any increasing and right-continuous function, and note that $\overline{v}(q)D_q(\Gamma) = \int_0^{\overline{v}(q)} \Gamma$ for every $q \in [0,1]$. Given $\hat{q} \in [0,1]$, let us now show Γ is greedy up to $\overline{v}(\hat{q})$—or equivalently, has $\int_0^{\overline{v}(q)} \Gamma = q\overline{v}(q)$ for every $q \in [0,\hat{q}]$—if and only if Γ agrees with Γ^* on $[0,\overline{v}(\hat{q}))$. To that end, note Lemma 7 tells us Γ can be greedy up to $\overline{v}(\hat{q})$ only if it is continuous on $[0,\overline{v}(\hat{q}))$ with $\Gamma(0) = 0$, so we can focus on such Γ. Because the equality $\int_0^{\overline{v}(q)} \Gamma = q\overline{v}(q)$ holds for $q = 0$ and both sides are differentiable in q, it holds for every $q \in (0,\hat{q})$ if and only if the derivatives coincide at every $q \in (0,\hat{q})$—that is

$$q\overline{v}'(q) + \overline{v}(q) = \overline{v}'(q)\Gamma\left(\overline{v}(q)\right).$$

Rearranging, Γ is greedy if and only if it agrees with Γ^* on $[0,\overline{v}(\hat{q}))$.

In particular, the equivalence of the previous paragraph tells us Γ^* is greedy, and Lemma 7 says it is strictly increasing on $[0,\overline{v}(1))$. Now, because Γ^* is strictly increasing, at most one $\overline{p}^* \in (0,\overline{v}(1))$ can exist with $\Gamma^*(\overline{p}^*) = 1$. Because Γ^* is continuous and

$$\Gamma^*(0) = 0 < 1 < 1 + \tfrac{\overline{v}(1)}{\overline{v}'(1)} = \Gamma^*(\overline{v}(1)),$$

the intermediate value theorem tells us some such $\overline{p}^*$ exists.

Observe next, because Γ^* is strictly increasing, it follows that the function $[0,\overline{v}(1)] \to \mathbb{R}$ given by $p \mapsto \int_0^p (1-\Gamma^*)$ is continuous and strictly concave and is maximized at $\overline{p}^*$. Moreover, its value at the right endpoint of its domain is $\int_0^{\overline{v}(1)} (1-\Gamma^*) = \overline{v}(1) - \int_0^{\overline{v}(1)} \Gamma^* = \overline{v}(1) - 1\overline{v}(1) = 0$, the same as its value at the left endpoint. Therefore, every $\hat{p} \in [0,\overline{p}^*]$ admits a unique $\hat{p}' \in [\overline{p}^*,\overline{v}(1)]$ such that $\int_0^{\hat{p}'} (1-\Gamma^*) = \int_0^{\hat{p}} (1-\Gamma^*)$—hence a unique $\hat{q} \in [\underline{q}(\overline{p}^*),1]$ such that $\int_{\hat{p}}^{\overline{v}(\hat{q})} (1-$

[39] Any such function is easily seen to be a limit in $C^1[0,1]$ of such functions that are also twice differentiable, and the induced $\Gamma^* \circ \overline{v}$ is then a limit of those for the approximating models, hence is weakly increasing.

$\Gamma^*) = 0$. Moreover, this $\hat{p}'$ continuously strictly decreases as $\hat{p}$ increases, and so too does $\hat{q}$. *Q.E.D.*

In line with the previous lemma, we can introduce the following notations:

Notation 2. *In the linear demand environment:*

(i) Define $\mathcal{Q} : [0, \overline{p}^*] \to [\underline{q}(\overline{p}^*), 1]$ *to be the unique function mapping any* $\hat{p} \in [0, \overline{p}^*]$ *to the unique* $\hat{q} \in [\underline{q}(\overline{p}^*), 1]$ *such that* $\int_{\hat{p}}^{\overline{v}(\hat{q})} (1 - \Gamma^*) = 0$.

(ii) Define $\mathcal{P} := \mathcal{Q}^{-1} : [\underline{q}(\overline{p}^*), 1] \to [0, \overline{p}^*]$ *and* $\mathcal{V} := v \circ \mathcal{Q} : [0, \overline{p}^*] \to [\overline{p}^*, \overline{v}(1)]$.

(iii) For each $\hat{p} \in [0, \overline{p}^*]$*, define* $\Pi(\cdot|\hat{p}) \in \Delta(\mathbb{R}_+)$ *via*

$$\Pi(p|\hat{p}) := \begin{cases} \Gamma^*(p) & : \ p < \hat{p} \\ 1 & : \ p \geq \hat{p}. \end{cases}$$

(iv) Define $\mathcal{R} : [0, \overline{p}^*] \times [\underline{q}(\overline{p}^*), 1] \to \mathbb{R}$ *by* $\mathcal{R}(\hat{p}, \hat{q}) := R_{\hat{q}}(\Pi_{\hat{p}})$.

Now, let us record some useful computations about these objects.

Lemma 9. *In the linear demand environment:*

(i) The functions $\mathcal{V}$ *and* $\mathcal{Q}$ *are continuously differentiable on* $[0, \overline{p}^*)$*, and* $\mathcal{P}$ *is continuously differentiable on* $(\underline{q}(\overline{p}^*), 1]$*. Any* $\hat{p} \in [0, \overline{p}^*)$ *and* $\hat{q} = \mathcal{Q}(\hat{p}) \in (\underline{q}(\overline{p}^*), 1]$ *have*

$$\mathcal{V}'(\hat{p}) = -\frac{1 - \Gamma^*(\hat{p})}{\Gamma^*(\overline{v}(\hat{q})) - 1}, \ \ \mathcal{Q}'(\hat{p}) = \frac{\mathcal{V}'(\hat{p})}{\overline{v}'(\hat{q})}, \ \ \text{and } \mathcal{P}'(\hat{q}) = \frac{1}{\mathcal{Q}'(\hat{p})},$$

which are all strictly negative.

(ii) Any $\hat{p} \in [0, \overline{p}^*]$ *has*

$$\int_0^{\infty} p^2 \ \mathrm{d}\Pi(p|\hat{p}) = 2 \int_0^{\hat{p}} p[1 - \Gamma^*(p)] \ \mathrm{d}p = \hat{p}^2 \left[1 - \underline{q}(\hat{p})\right] - \int_0^{\underline{q}(\hat{p})} \overline{v}^2.$$

(iii) Any $\hat{p} \in [0, \overline{p}^*]$ *and* $\hat{q} \in [\underline{q}(\overline{p}^*), 1]$ *have*

$$\mathcal{R}(\hat{p}, \hat{q}) = \int_0^{\hat{p}} \left[1 - \tfrac{2p}{\overline{v}(\hat{q})}\right] [1 - \Gamma^*(p)] \ \mathrm{d}p.$$

(iv) *Any* $\hat{p} \in [0, \overline{p}^*)$ *has* $\frac{\mathrm{d}}{\mathrm{d}\hat{p}}\mathcal{R}(\hat{p}, \mathcal{Q}(\hat{p})) = \frac{1-\Gamma^*(\hat{p})}{\mathcal{V}(\hat{p})} r(\hat{p})$*, where*

$$r(\hat{p}) := [\mathcal{V}(\hat{p}) - 2\hat{p}] - \frac{2}{\mathcal{V}(\hat{p})\,[\Gamma^*(\mathcal{V}(\hat{p})) - 1]} \int_0^{\hat{p}} p\,[1 - \Gamma^*(p)]\ \mathrm{d}p.$$

(v) *The function* $r : [0, \overline{p}^*) \to \mathbb{R}$ *is continuously differentiable with strictly negative derivative.*

(vi) *The function* $[0, \overline{p}^*] \to \mathbb{R}$ *given by* $\hat{p} \mapsto \mathcal{R}(\hat{p}, \mathcal{Q}(\hat{p}))$ *is strictly quasiconcave with interior maximum.*

Proof. We first establish the derivative computations for $\mathcal{V}$, $\mathcal{Q}$, and $\mathcal{P}$. We need only show the given properties for $\mathcal{V}$, and then those for $\mathcal{Q}$ and $\mathcal{P}$ follow directly from the chain rule. At any $\hat{p} \in [0, \overline{p}^*)$, that $\mathcal{Q}(\hat{p}) > \underline{q}(\overline{p}^*)$ implies the second partial derivative of the continuously differentiable function $(p, v) \mapsto \int_p^v (1 - \Gamma^*)$ is nonzero at $(\hat{p}, \mathcal{V}(\hat{p}))$. The implicit function theorem therefore implies $\mathcal{V}$ is differentiable at $\hat{p}$ with

$$0 = \frac{\mathrm{d}}{\mathrm{d}\hat{p}} \int_{\hat{p}}^{\mathcal{V}(\hat{p})} (1 - \Gamma^*) = \mathcal{V}'(\hat{p})\,[1 - \Gamma^*(\mathcal{V}(\hat{p}))] - [1 - \Gamma^*(\hat{p})]\,.$$

Thus, $\mathcal{V}'$ is as desired.

Next, observe that the expectation of the squared price given $\Pi(\cdot|\hat{p})$ is

$$\begin{aligned}
\int_0^\infty p^2\ \mathrm{d}\Pi(p|\hat{p}) &= [1 - \Gamma^*(\hat{p})]\hat{p}^2 + \int_0^{\hat{p}} p^2\ \mathrm{d}\Gamma^*(p) \\
&= \hat{p}^2 - \hat{p}^2\Gamma^*(\hat{p}) + \left[p^2\Gamma^*(p)\right]_{p=0}^{\hat{p}} - \int_0^{\hat{p}} 2p\Gamma^*(p)\ \mathrm{d}p \\
&= \hat{p}^2 - 2\int_0^{\hat{p}} p\Gamma^*(p)\ \mathrm{d}p \\
&= 2\int_0^{\hat{p}} p\,[1 - \Gamma^*(p)]\ \mathrm{d}p,
\end{aligned}$$

which is in turn equal to $\hat{p}^2\left[1 - \underline{q}(\hat{p})\right] - \int_0^{\underline{q}(\hat{p})} \overline{v}^2$ because the two expressions are both zero for $\hat{p} = 0$ and have the same derivative with respect to $\hat{p}$.

Toward computing $\mathcal{R}(\hat{p}, \hat{q})$, note that

$$\int_0^\infty p \, \mathrm{d}\Pi(p|\hat{p}) = \int_0^\infty [1 - \Pi(\cdot|\hat{p})] = \int_0^{\hat{p}} (1 - \Gamma^*).$$

Moreover, that $\hat{p} \in [0, \overline{p}^*]$ and $\hat{q} \in [\underline{q}(\overline{p}^*), 1]$ implies $\hat{p} \leq \overline{v}(\hat{q})$. Thus,

$$\begin{aligned}
\overline{v}(\hat{q})\mathcal{R}(\hat{p}, \hat{q}) &= \overline{v}(\hat{q}) \int_0^\infty p \left[1 - \tfrac{p}{\overline{v}(\hat{q})}\right] \mathrm{d}\Pi(p|\hat{p}) \\
&= \overline{v}(\hat{q}) \int_0^\infty p \, \mathrm{d}\Pi(p|\hat{p}) - \int_0^\infty p^2 \, \mathrm{d}\Pi(p|\hat{p}) \\
&= \overline{v}(\hat{q}) \int_0^{\hat{p}} (1 - \Gamma^*) - 2 \int_0^{\hat{p}} p \, [1 - \Gamma^*(p)] \, \mathrm{d}p.
\end{aligned}$$

Hence, $\mathcal{R}(\hat{p}, \hat{q}) = \int_0^{\hat{p}} \left[1 - \frac{2p}{\overline{v}(\hat{q})}\right] [1 - \Gamma^*(p)] \, \mathrm{d}p$.

Now, because

$$\begin{aligned}
\frac{\partial}{\partial \hat{q}}\bigg|_{\hat{q}=\mathcal{Q}(\hat{p})} \mathcal{R}(\hat{p}, \hat{q}) &= \left\{ \int_0^{\hat{p}} 2p \, [1 - \Gamma^*(p)] \; \mathrm{d}p \right\} \frac{\partial}{\partial \hat{q}}\bigg|_{\hat{q}=\mathcal{Q}(\hat{p})} \left[\frac{-1}{\overline{v}(\hat{q})}\right] \\
&= \frac{2\overline{v}'(\hat{q})}{\overline{v}(\hat{q})^2} \int_0^{\hat{p}} p \, [1 - \Gamma^*(p)] \; \mathrm{d}p,
\end{aligned}$$

the chain rule yields

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\hat{p}} \mathcal{R}(\hat{p}, \mathcal{Q}(\hat{p})) &= \frac{\partial}{\partial \hat{p}} \mathcal{R}(\hat{p}, \mathcal{Q}(\hat{p})) + \mathcal{Q}'(\hat{p}) \, \frac{\partial}{\partial \hat{q}}\bigg|_{\hat{q}=\mathcal{Q}(\hat{p})} \mathcal{R}(\hat{p}, \hat{q}) \\
&= \left[1 - \frac{2\hat{p}}{\overline{v}(\mathcal{Q}(\hat{p}))}\right] [1 - \Gamma^*(\hat{p})] + \frac{\mathcal{V}'(\hat{p})}{\overline{v}'(\mathcal{Q}(\hat{p}))} \frac{2\overline{v}'(\mathcal{Q}(\hat{p}))}{\overline{v}(\mathcal{Q}(\hat{p}))^2} \int_0^{\hat{p}} p \, [1 - \Gamma^*(p)] \; \mathrm{d}p \\
&= \frac{1 - \Gamma^*(\hat{p})}{\overline{v}(\mathcal{Q}(\hat{p}))} \left\{ [\overline{v}(\mathcal{Q}(\hat{p})) - 2\hat{p}] + \frac{2\mathcal{V}'(\hat{p})}{\overline{v}(\mathcal{Q}(\hat{p})) \, [1 - \Gamma^*(\hat{p})]} \int_0^{\hat{p}} p \, [1 - \Gamma^*(p)] \; \mathrm{d}p \right\} \\
&= \frac{1 - \Gamma^*(\hat{p})}{\overline{v}(\mathcal{Q}(\hat{p}))} r(\hat{p}).
\end{aligned}$$

Next, that r is continuously differentiable on $[0, \overline{p}^*)$ follows directly from $\mathcal{V}$ being so and Γ^* being continuous. To see r has strictly negative derivative on $[0, \overline{p}^*)$, it suffices to see that $r(\hat{p}) + 2\hat{p}$ is decreasing on this range. And indeed, because $\mathcal{V}$ is decreasing there, it follows that $\hat{p} \mapsto r(\hat{p}) + 2\hat{p}$ is a decreasing function minus the ratio of a positive increasing function to a positive decreasing function—and hence is decreasing as desired.

Finally, because $\frac{\mathrm{d}}{\mathrm{d}\hat{p}}\mathcal{R}\left(\hat{p}, \mathcal{Q}(\hat{p})\right)$ is a strictly positive multiple of $r(\hat{p})$, which is strictly decreasing in $\hat{p} \in [0, \overline{p}^*)$, it follows that $\hat{p} \mapsto \mathcal{R}\left(\hat{p}, \mathcal{Q}(\hat{p})\right)$ is strictly quasiconcave on $[0, \overline{p}^*)$—hence on $[0, \overline{p}^*]$ by continuity. Moreover, $\hat{p} \mapsto \mathcal{R}\left(\hat{p}, \mathcal{Q}(\hat{p})\right)$ is maximized on the interior of its domain if r has an interior root. And indeed, $r(0) = \overline{v}(1) > 0$, whereas any $\hat{p} \in [0, \overline{p}^*)$ close enough to $\overline{p}^*$ has $\mathcal{V}(\hat{p}) < 2\hat{p}$ and so $r(\hat{p}) < 0$. Therefore, r has an interior root by the intermediate value theorem. *Q.E.D.*

Lemma 10. *In the linear demand environment, (Π^*, q^*) is optimal if and only if $\Pi^* = \Pi(\cdot|p^*)$ and $q^* = \mathcal{Q}(p^*)$ for the unique $p^* \in (0, \overline{p}^*)$ satisfying $r(p^*) = 0$.*

Proof. First, we observe any optimal (Π^*, q^*) must have $\Pi^* = \Pi(\cdot|\hat{p})$ for some $\hat{p} \in [0, \overline{p}^*]$. To see this, note that Theorem 1 tells us Π^* is greedy up to the top of its support p^*. But then Lemma 8 tells us Π^* agrees with Γ^* on $[0, p^*)$, and so fact that Π^* is in $\Delta(\mathbb{R}_+)$ tells us $\Pi^* = \Pi(\cdot|p^*)$ and $p^* \leq \overline{p}^*$.

Now we argue that, given $\hat{p} \in [0, \overline{p}^*]$, the set of all $q \in (0, 1]$ with $D_q(\Pi(\cdot|\hat{p})) \geq q$ is equal to $[0, \mathcal{Q}(\hat{p})]$. Toward this characterization, first note (given Lemma 8) any $q \in [0, \underline{q}(\hat{p})]$ has $D_q(\Pi(\cdot|\hat{p})) = \int_0^{\overline{v}(q)} \Pi(\cdot|\hat{p}) f_q = \int_0^{\overline{v}(q)} \Gamma^* f_q = D_q(\Gamma^*) = q$. Next observe, any $q \in [\underline{q}(\hat{p}), 1]$ has (again by Lemma 8)

$$\overline{v}(q) D_q(\Pi(\cdot|\hat{p})) = \overline{v}(q) \int_0^{\overline{v}(q)} \Pi(\cdot|\hat{p}) f_q = \int_0^{\overline{v}(q)} \Pi(\cdot|\hat{p}) = \int_0^{\hat{p}} \Gamma^* + \int_{\hat{p}}^{\overline{v}(q)} 1,$$

and so $\overline{v}(q)\left[D_q(\Pi(\cdot|\hat{p})) - q\right] = \int_0^{\hat{p}} \Gamma^* + \int_{\hat{p}}^{\overline{v}(q)} 1 - \int_0^{\overline{v}(q)} \Gamma^* = \int_{\hat{p}}^{\overline{v}(q)} (1 - \Gamma^*)$. Therefore, because Lemma 8 says Γ^* is strictly increasing, the function $q \mapsto \overline{v}(q)\left[D_q(\Pi(\cdot|\hat{p})) - q\right]$ is strictly quasiconcave on $[\underline{q}(\hat{p}), 1]$ and zero at $\underline{q}(\hat{p})$. Because the function also takes value zero at $\mathcal{Q}(\hat{p})$, it is then nonnegative up to $\mathcal{Q}(\hat{p})$ and strictly negative to the right. Thus, $\{q \in (0, 1] : \; D_q(\Pi(\cdot|\hat{p})) \geq q\} = [0, \mathcal{Q}(\hat{p})]$, as desired.

By the previous two paragraphs, we can write the seller's problem (P*) as

$$\max_{\hat{p}\in[0,\overline{p}^*],\ \hat{q}\in[0,1]} R_{\hat{q}}(\Pi(\cdot|\hat{p})) \text{ s.t. } \hat{q} \le \mathcal{Q}(\hat{p}).$$

Because Lemma 2 tells us the objective is strictly increasing (wherever strictly positive, as the optimal revenue is) in the quantity, the seller optimally sets $\hat{q} = \mathcal{Q}(\hat{p})$, and so her problem can be written as

$$\max_{\hat{p}\in[0,\overline{p}^*]} \mathcal{R}(\hat{p}, \mathcal{Q}(\hat{p})).$$

By Lemma 9, this objective is strictly quasiconcave with interior optimum, and the optimum p^* is characterized by $r(p^*) = 0$. *Q.E.D.*

C.2. Proof of Proposition 3

We begin by proving the following Lemma 11, which we will then use in the proof of Proposition 3.

Lemma 11. *Take the linear demand environment.*

(i) If the seller posts a price strictly greater than $\overline{p}^B := \int_0^{\overline{p}^*}(1-\Gamma^*) \in (0, \overline{p}^*)$, *then the highest equilibrium quantity is zero.*

(ii) If the seller posts a price $\hat{p} \in [0, \overline{p}^B]$, *then a given* $\hat{q}$ *is an equilibrium quantity if and only if* $\hat{p} = \mathcal{P}^B(\hat{q})$, *where* $\mathcal{P}^B(\hat{q}) := \int_0^{\overline{v}(\hat{q})}(1-\Gamma^*)$. *In particular, the highest such quantity is the unique* $\mathcal{Q}^B(\hat{p}) \in [\underline{q}(\overline{p}^*), 1]$ *such that* $\mathcal{P}^B(\mathcal{Q}^B(\hat{p})) = \hat{p}$.

(iii) The functions $\mathcal{P}^B$ *and* $\mathcal{Q}^B$ *are continuously differentiable on* $[0, \overline{p}^B)$ *and* $(\underline{q}(\overline{p}^*), 1]$, *respectively. Any* $\hat{p} \in [0, \overline{p}^*)$ *and* $\hat{q} = \mathcal{Q}(\hat{p}) \in (\underline{q}(\overline{p}^*), 1]$ *have*

$$\frac{\mathrm{d}}{\mathrm{d}\hat{q}}\mathcal{P}^B(\hat{q}) = -\overline{v}'(\hat{q})\,[\Gamma^*(\overline{v}(\hat{q})) - 1] \text{ and } \frac{\mathrm{d}}{\mathrm{d}\hat{p}}\mathcal{Q}^B(\hat{p}) = \frac{1}{\frac{\mathrm{d}}{\mathrm{d}\hat{q}}\big|_{\hat{q}=\mathcal{Q}^B(\hat{p})}\mathcal{P}^B(\hat{q})},$$

which are both strictly negative.

(iv) Letting $\mathcal{R}^B(\hat{p}, \hat{q}) := R_{\hat{q}}(\hat{p})$, *any* $\hat{p} \in [0, \overline{p}^B)$ *has* $\frac{\mathrm{d}}{\mathrm{d}\hat{p}}\mathcal{R}^B\left(\hat{p}, \mathcal{Q}^B(\hat{p})\right) =$

$\frac{1}{\overline{v}(\mathcal{Q}^B(\hat{p}))} r^B(\hat{p})$, *where*

$$r^B(\hat{p}) := \left[\overline{v}(\mathcal{Q}^B(\hat{p})) - 2\hat{p}\right] - \frac{1}{\overline{v}(\mathcal{Q}^B(\hat{p}))\left[\Gamma^*(\overline{v}(\mathcal{Q}^B(\hat{p}))) - 1\right]}\,\hat{p}^2.$$

(v) *The function $r^B : [0, \overline{p}^B) \to \mathbb{R}$ is continuously differentiable with strictly negative derivative.*

(vi) *The function $[0, \overline{p}^B] \to \mathbb{R}$ given by $\hat{p} \mapsto \mathcal{R}^B\left(\hat{p}, \mathcal{Q}^B(\hat{p})\right)$ is strictly quasiconcave with interior maximum.*

(vii) *Under best-case equilibrium selection, the unique optimal price distribution is degenerate at the unique price $p^B \in (0, \overline{p}^*)$ with $r^B(p^B) = 0$, and the unique best equilibrium quantity at that price is $q^B := \mathcal{Q}^B(p^B)$.*

Proof. By Proposition 1, the seller optimally chooses a deterministic price and so solves

$$\max_{\hat{p}\in\mathbb{R}_+,\ \hat{q}\in[0,1]} \mathcal{R}^B(\hat{p}, \hat{q}) \text{ s.t. } D_{\hat{q}}(\hat{p}) = \hat{q}.$$

By the same proposition, an optimum exists and has both price and quantity being strictly positive. Now, let us rewrite the equilibrium constraint. Any quantity $\hat{q} \in (0, 1]$ and price $\hat{p} \in \mathbb{R}_+$ have

$$D_{\hat{q}}(\hat{p}) = \hat{q} \iff 1 - \tfrac{\hat{p}}{\overline{v}(\hat{q})} = \hat{q} \iff \hat{p} = (1 - \hat{q})\overline{v}(\hat{q}) \iff \hat{p} = \int_0^{\overline{v}(\hat{q})} (1 - \Gamma^*),$$

where the last equivalence follows from Lemma 8. Now, defining $\mathcal{P}^B : [0, 1] \to \mathbb{R}$ given by $\mathcal{P}^B(\hat{q}) := \int_0^{\overline{v}(\hat{q})}(1 - \Gamma^*)$, Lemma 8 tells us $\mathcal{P}^B$ is continuous and strictly quasiconcave with $\mathcal{P}^B(0) = \mathcal{P}^B(1) = 0$ and maximizer $\underline{q}(\overline{p}^*)$. Therefore, the range of $\mathcal{P}^B$ is $[0, \overline{p}^B]$ for $\overline{p}^B := \int_0^{\overline{p}^*}(1 - \Gamma^*) \in (0, \overline{p}^*)$, and every $\hat{p} \in [0, \overline{p}^B]$ has one solution in $[0, \underline{q}(\overline{p}^*)]$ and one solution in $[\underline{q}(\overline{p}^*), 1]$ to $\mathcal{P}^B(\cdot) = \hat{p}$. So let $\mathcal{Q}^B : [0, \overline{p}^B] \to [\underline{q}(\overline{p}^*), 1]$ be such that $\mathcal{P}^B(\mathcal{Q}^B(\hat{p})) = \hat{p}$ for every $\hat{p} \in [0, \overline{p}^B]$; Lemma 2 tells us seller revenue is strictly increasing (wherever strictly positive, as the optimal revenue is) in the quantity, and so the best equilibrium quantity if the seller set price $\hat{p}$ is $\hat{q} = \mathcal{Q}^B(\hat{p})$. We can thus write the seller's problem under best case selection as

$$\max_{\hat{p}\in[0,\overline{p}^B]} \mathcal{R}^B(\hat{p}, \mathcal{Q}^B(\hat{p})).$$

Now, the function $\mathcal{Q}^B$ is continuous and strictly decreasing by construction. Moreover, because $\frac{\mathrm{d}}{\mathrm{d}\hat{q}}\mathcal{P}^B(\hat{q}) = -\overline{v}'(\hat{q})\,[\Gamma^*(\overline{v}(\hat{q})) - 1]$, which is continuous and strictly negative for $\hat{q} \in [0, \underline{q}(\overline{p}^B))$, the inverse function theorem tells us $\mathcal{Q}^B$ is continuously differentiable on $(0, \overline{p}^B]$ with derivative $\frac{\mathrm{d}}{\mathrm{d}\hat{p}}\mathcal{Q}^B(\hat{p}) = \frac{1}{\overline{v}'(\mathcal{Q}^B(\hat{p}))[\Gamma^*\circ\overline{v}\circ\mathcal{Q}^B(\hat{p})-1]}$ there. We are now equipped to compute the seller's first-order condition under best-case selection. For any $\hat{p} \in (0, \overline{p}^B)$, at $\hat{q} = \mathcal{Q}^B(\hat{p})$ and $\hat{v} = \overline{v}(\hat{q})$ we have

$$\begin{aligned}
\overline{v}(\mathcal{Q}^B(\hat{p}))\,\frac{\mathrm{d}}{\mathrm{d}\hat{p}}\mathcal{R}^B(\hat{p}, \mathcal{Q}^B(\hat{p})) &= \hat{v}\left[\frac{\partial}{\partial\hat{p}}\mathcal{R}^B(\hat{p}, \hat{q}) + (\mathcal{Q}^B)'(\hat{p})\,\frac{\partial}{\partial\hat{q}}\mathcal{R}^B(\hat{p}, \hat{q})\right] \\
&= \hat{v}\left\{\frac{\partial}{\partial\hat{p}}\left[\hat{p}\left(1 - \frac{\hat{p}}{\overline{v}(\hat{q})}\right)\right] + (\mathcal{Q}^B)'(\hat{p})\,\frac{\partial}{\partial\hat{q}}\left[\hat{p}\left(1 - \frac{\hat{p}}{\overline{v}(\hat{q})}\right)\right]\right\} \\
&= \hat{v}\left\{\left[1 - \frac{2\hat{p}}{\hat{v}}\right] + \frac{-1}{\overline{v}'(\hat{q})\,[\Gamma^*(\hat{v}) - 1]}\,\frac{\hat{p}^2}{\hat{v}^2}\overline{v}'(\hat{q})\right\} \\
&= [\hat{v} - 2\hat{p}] - \frac{1}{\hat{v}\,[\Gamma^*(\hat{v}) - 1]}\,\hat{p}^2 \\
&= \left[\overline{v}(\mathcal{Q}^B(\hat{p})) - 2\hat{p}\right] - \frac{1}{\overline{v}(\mathcal{Q}^B(\hat{p}))\,[\Gamma^*(\overline{v}(\mathcal{Q}^B(\hat{p}))) - 1]}\,\hat{p}^2,
\end{aligned}$$

which is $r^B(\hat{p})$. Because the denominator $\overline{v}(\mathcal{Q}^B(\hat{p}))\left[\Gamma^*(\overline{v}(\mathcal{Q}^B(\hat{p}))) - 1\right]$ is positive and decreasing in $\hat{p} \in (0, \overline{p}^B)$, it follows that the decreasing r^B has strictly negative derivative on $(0, \overline{p}^B)$—hence the seller's problem under best-case selection is strictly quasiconcave in the choice of posted price. To see an interior root of r^B exists, note that $r(0) = \overline{v}(1) > 0$, whereas any $\hat{p} \in (0, \overline{p}^B)$ close enough to $\overline{p}^B$ has $v(\mathcal{Q}^B(\hat{p})) < 2\hat{p}$ and so $r^B(\hat{p}) < 0$; thus the intermediate value theorem applies. *Q.E.D.*

Proof of Proposition 3. Given Lemma 11, we know the strictly positive price $p^B < \overline{p}^B < \overline{p}^*$ is the unique price such that $r^B(p^B) = 0$, and $q^B = \mathcal{Q}^B(p^B)$. We will use these facts to compare with worst-case selection.

Let p^* and q^* be the highest supported price and equilibrium quantity as described in Lemma 10. We want to show that $p^* > p^B$ and $q^* > q^B$. Because $\mathcal{Q}$ is strictly decreasing, we can equivalently show that $\mathcal{P}(q^B) > p^* > p^B$. By Lemma 9, we can rewrite this condition as the requirement that $r(\mathcal{P}(q^B)) < 0 < r(p^B)$. Toward both of these rankings, observe that any $\hat{p} \in (0, \overline{p}^*)$ has

$$\begin{aligned} r(\hat{p}) &= r(\hat{p}) - r^B(p^B) \\ &= [\mathcal{V}(\hat{p}) - 2\hat{p}] - \left[\overline{v}(q^B) - 2p^B\right] \\ &\quad - \tfrac{1}{\mathcal{V}(\hat{p})[\Gamma^*(\mathcal{V}(\hat{p}))-1]} \int_0^\infty p^2 \, \mathrm{d}\Pi(p|\hat{p}) + \tfrac{1}{\overline{v}(q^B)[\Gamma^*(\overline{v}(q^B))-1]} \, (p^B)^2 \end{aligned}$$

Toward the price ranking, note that specializing the above calculation yields

$$r(p^B) = \mathcal{V}(p^B) - \overline{v}(q^B) - \tfrac{1}{\mathcal{V}(p^B)[\Gamma^*(\mathcal{V}(p^B))-1]} \int_0^\infty p^2 \, \mathrm{d}\Pi(p|p^B) + \tfrac{1}{\overline{v}(q^B)[\Gamma^*(\overline{v}(q^B))-1]} \, (p^B)^2.$$

That $\Pi(p^B|p^B) = 1$ implies $\int_0^\infty p^2 \, \mathrm{d}\Pi(p|p^B) \leq (p^B)^2$; and that Γ^* is increasing (Lemma 8) and $\mathcal{V}$ decreasing implies $\frac{1}{\mathcal{V}\cdot[\Gamma^* \circ \mathcal{V} - 1]}$ is increasing on $[0, \overline{p}^*)$. Hence, $r(p^B) > 0$ will follow if we show $\mathcal{V}(p^B) > \overline{v}(q^B)$. And indeed,

$$\begin{aligned} \int_{\overline{v}(q^B)}^{\mathcal{V}(p^B)} (\Gamma^* - 1) &= \left(\int_0^{\overline{v}(q^B)} - \int_{p^B}^{\mathcal{V}(p^B)} - \int_0^{p^B} \right) (1 - \Gamma^*) \\ &= p^B - 0 - \int_0^{p^B} (1 - \Gamma^*) \\ &= \int_0^{p^B} \Gamma^* > 0, \end{aligned}$$

delivering $\mathcal{V}(p^B) > \overline{v}(q^B)$ because $\Gamma^* > 1$ between them. The price ranking follows.

Toward the quantity ranking, observe that

$$\begin{aligned} r(\mathcal{P}(q^B)) &= \left[\mathcal{V}(\mathcal{P}(q^B)) - 2\mathcal{P}(q^B)\right] - \left[\overline{v}(q^B) - 2p^B\right] \\ &\quad - \tfrac{1}{\mathcal{V}(\mathcal{P}(q^B))[\Gamma^*(\mathcal{V}(\mathcal{P}(q^B)))-1]} \int_0^\infty p^2 \, \mathrm{d}\Pi(p|\mathcal{P}(q^B)) + \tfrac{1}{\overline{v}(q^B)[\Gamma^*(\overline{v}(q^B))-1]} \, (p^B)^2 \\ &= -2\left[\mathcal{P}(q^B) - p^B\right] - \tfrac{1}{\overline{v}(q^B)[\Gamma^*(\overline{v}(q^B))-1]} \left[\int_0^\infty p^2 \, \mathrm{d}\Pi(p|\mathcal{P}(q^B)) - (p^B)^2 \right]. \end{aligned}$$

So $r(\mathcal{P}(q^B)) < 0$ would follow if we knew $\mathcal{P}(q^B) > p^B$ and $\int_0^\infty p^2 \,\mathrm{d}\Pi(p|\mathcal{P}(q^B)) \geq (p^B)^2$. To establish both inequalities, observe that

$$p^B = p^B - 0 = \left(\int_0^{\overline{v}(q^B)} - \int_{\mathcal{P}(q^B)}^{\mathcal{V}(\mathcal{P}(q^B))}\right)(1-\Gamma^*) = \int_0^{\mathcal{P}(q^B)}(1-\Gamma^*).$$

This identity first implies $\mathcal{P}(q^B) > p^B$ because $\int_0^{\mathcal{P}(q^B)}(1-\Gamma^*) < \mathcal{P}(q^B)$. Then, to show $\int_0^\infty p^2 \,\mathrm{d}\Pi(p|\mathcal{P}(q^B)) \geq (p^B)^2$, it suffices to show $\int_0^\infty p^2 \,\mathrm{d}\Pi(p|\hat{p}) - \left[\int_0^{\hat{p}}(1-\Gamma^*)\right]^2$ is nonnegative for any $\hat{p} \in [0, \overline{p}^*]$. And indeed, the expression is obviously zero for $\hat{p} = 0$, and it satisfies

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\hat{p}}\left\{\int_0^\infty p^2 \,\mathrm{d}\Pi(p|\hat{p}) - \left[\int_0^{\hat{p}}(1-\Gamma^*)\right]^2\right\} &= \frac{\mathrm{d}}{\mathrm{d}\hat{p}}\left\{2\int_0^{\hat{p}} p\left[1-\Gamma^*(p)\right] \,\mathrm{d}p - \left[\int_0^{\hat{p}}(1-\Gamma^*)\right]^2\right\} \\
&= 2\hat{p}\left[1-\Gamma^*(\hat{p})\right] - 2\left[\int_0^{\hat{p}}(1-\Gamma^*)\right]\left[1-\Gamma^*(\hat{p})\right] \\
&= 2\left[1-\Gamma^*(\hat{p})\right]\int_0^{\hat{p}}\Gamma^* \geq 0.
\end{aligned}$$

The quantity ranking follows.

Finally, we turn to the consumer surplus ranking. Let $\Pi^* := \Pi(\cdot|p^*)$ be the optimal (under worst-case selection) price distribution, and let Π be the modified price distribution given by capping the price at $\overline{v}(q^B)$—that is $\Pi(p)$ is equal to $\Pi^*(p)$ for $p < \overline{v}(q^B)$, and is equal to 1 for $p \geq \overline{v}(q^B)$. Observe, any $p \in \mathbb{R}_+$ has

$$\begin{aligned}
\tfrac{1}{\overline{v}(q^B)}\left[p^B - \min\{p,\ \overline{v}(q^B)\}\right] &= D_{q^B}(\min\{p,\ \overline{v}(q^B)\}) - D_{q^B}(p^B) \\
&= D_{q^B}(p) - q^B,
\end{aligned}$$

and so $\frac{1}{\overline{v}(q^B)}\left[p^B - \int_0^\infty p \,\mathrm{d}\Pi(p)\right] = D_{q^B}(\Pi^*) - q^B$, which is nonnegative because $q^B < q^*$ and (Π^*, q^*) satisfies the demand constraints. Having established $p^B \geq \int_0^\infty p \,\mathrm{d}\Pi(p)$, we now pursue the surplus ranking. To that end, define

$$\mathrm{CS}_q(p) := \int (v-p)_+ \,\mathrm{d}F_q(v) = \int_p^\infty D_q$$

the consumer surplus associated with anticipated quantity q (hence demand curve D_q) and a price offer of p; and let $\mathrm{CS}_q(\hat{\Pi}) := \int \mathrm{CS}_q(p)\ \mathrm{d}\hat{\Pi}(p)$ for any price distribution $\hat{\Pi}$. Observe that $\mathrm{CS}_q(p)$ is decreasing in p, strictly convex in p (because D_q is strictly decreasing) wherever $0 \le p \le \overline{v}(q)$, and (because $u(\theta, \cdot)$ is increasing) increasing in q. Moreover, the price distribution Π is nondegenerate because $p^B > 0$ and Π^* has nondegenerate convex support including zero. Therefore,

$$\mathrm{CS}_{q^B}(p^B) \le \mathrm{CS}_{q^B}\left(\int p\ \mathrm{d}\Pi(p)\right) < \mathrm{CS}_{q^B}(\Pi) = \mathrm{CS}_{q^B}(\Pi^*) \le \mathrm{CS}_{q^*}(\Pi^*),$$

where the last inequality holds because $q^* \ge q^B$. *Q.E.D.*

C.3. Proof of Proposition 4

For any $\omega \in [0,1]$, define $\overline{v}_\omega : [0,1] \to \mathbb{R}$ by letting $\overline{v}_\omega(q) := \frac{1}{(1-\omega)\frac{1}{\overline{v}_0(q)} + \omega\frac{1}{\overline{v}_1(q)}}$ for $q \in (0,1]$, and $\overline{v}_\omega(0) := 0$; this $\overline{v}_\omega$ is also an instance of the linear demand environment. In particular, $\frac{1}{\overline{v}_\omega}$ inherits strict convexity from $\frac{1}{\overline{v}_0}$ and $\frac{1}{\overline{v}_1}$. Observe that any $q \in (0,1]$ has

$$\frac{\partial}{\partial\omega}\log\overline{v}_\omega(q) = -\frac{\partial}{\partial\omega}\log\frac{1}{\overline{v}_\omega(q)} = -\frac{\frac{1}{\overline{v}_1(q)} - \frac{1}{\overline{v}_0(q)}}{\frac{1}{\overline{v}_\omega(q)}} = \frac{\frac{\overline{v}_1(q)}{\overline{v}_0(q)} - 1}{(1-\omega)\frac{\overline{v}_1(q)}{\overline{v}_0(q)} + \omega},$$

which is strictly increasing in q because $\frac{\overline{v}_1(q)}{\overline{v}_0(q)}$ is. Equivalently, whenever $0 < q < \tilde{q} \le 1$, we have $\frac{\partial}{\partial\omega}\left[\frac{\overline{v}_\omega(\tilde{q})}{\overline{v}_\omega(q)}\right] > 0$, a log-supermodularity property that will be useful in establishing the quantity ranking.

First, we pursue the price distribution ranking. That $\frac{\overline{v}_1}{\overline{v}_0}$ has nonnegative derivative on $(0,1]$ means $\frac{\overline{v}_1}{\overline{v}_1'} \le \frac{\overline{v}_0}{\overline{v}_0'}$, and so $\Gamma_1^* \circ \overline{v}_1 \le \Gamma_0^* \circ \overline{v}_0$. Using this fact, let us see that $\Gamma_1^*(p) < \Gamma_0^*(p)$ for any price p with $0 < p \le \min\{\overline{v}_0(1), \overline{v}_1(1)\}$. To see it, let $q_\omega := \overline{v}_\omega^{-1}(p) \in (0,1]$ for each $\omega \in \{0,1\}$. That $\overline{v}_1$ exhibits stronger externalities than $\overline{v}_0$ implies $q_1 < q_0$—since the strictly increasing function $\frac{\overline{v}_1}{\overline{v}_0}$

is above 1 on $(0,1]$, it is in fact strictly above 1 there.[40] It follows that

$$\Gamma_0^*(p) = \Gamma_0^*(\overline{v}_0(q_0)) \geq \Gamma_1^*(\overline{v}_1(q_0)) > \Gamma_1^*(\overline{v}_1(q_1)) = \Gamma_1^*(p).$$

Therefore, given Lemma 10, any p with $0 < p < \min\{p_1^*, p_0^*\}$ has $\Pi_0^*(p) = \Gamma_0^*(p) < \Gamma_1^*(p) = \Pi_1^*(p)$.

Next, we turn to the quantity ranking. Define $\tilde{r}(\hat{q},\omega) := \frac{r_\omega(\mathcal{P}_\omega(\hat{q}))}{\mathcal{P}_\omega(\hat{q})}$ for any $(\hat{q},\omega) \in [0,1) \times [0,1]$ with $\overline{v}_\omega(\hat{q}) > \overline{p}_\omega^*$. Below, we will show that the function $\tilde{r}$ has strictly negative partial derivative with respect to its second argument at (q_ω^*, ω) for any $\omega \in [0,1]$; let us now see that doing so would establish the result. First, an application of the implicit function theorem tells us $(\hat{q},\omega) \mapsto \mathcal{P}_\omega(\hat{q})$ is continuously differentiable on the range of $(\hat{q},\omega) \in [0,1) \times [0,1]$ with $\overline{v}_\omega(\hat{q}) > \overline{p}_\omega^*$, and that $\mathcal{P}_\omega$ is strictly decreasing (Lemma 9) tells us it is strictly positive there. Because $(\hat{p},\omega) \mapsto r_\omega(\hat{p})$ is continuously differentiable on the range of $(\hat{p},\omega) \in \mathbb{R}_+ \times [0,1]$ with $\hat{p} < \overline{p}_\omega^*$, it follows that $\tilde{r}$ is continuously differentiable on its domain. Next, observe, any $\omega \in [0,1]$ has

$$\frac{\partial}{\partial \hat{q}}\bigg|_{\hat{q}=q_\omega^*} \tilde{r}(\hat{q},\omega) = \frac{p_\omega^* r_\omega'(p_\omega^*) - 1 r_\omega(p_\omega^*)}{(p_\omega^*)^2}\mathcal{P}_\omega'(q_\omega^*) = \frac{\mathcal{P}_\omega'(q_\omega^*)}{p_\omega^*} r_\omega'(p_\omega^*) > 0,$$

where the second equality holds by Lemma 9 and Lemma 10 and the strict inequality follows from Lemma 9. Because q_ω^* is the unique solution $\hat{q}$ to $\tilde{r}(\hat{q},\omega) = 0$ for each $\omega \in [0,1]$, it follows that $\omega \mapsto q_\omega^*$ is continuously differentiable. Therefore, at any $\omega \in [0,1]$ and $\hat{q} = q_\omega^*$, we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}\omega} 0 = \frac{\mathrm{d}}{\mathrm{d}\omega}\tilde{r}(q_\omega^*, \omega) = \left[\frac{\partial}{\partial \omega}\tilde{r}(\hat{q},\omega)\right] + \left[\frac{\partial}{\partial \hat{q}}\tilde{r}(\hat{q},\omega)\right]\left[\frac{\partial}{\partial \omega} q_\omega^*\right].$$

Because we have shown $\frac{\partial}{\partial \hat{q}}\big|_{\hat{q}=q_\omega^*}\tilde{r}(\hat{q},\omega) > 0$, the hypothesis that $\frac{\partial}{\partial \omega}\big|_{\hat{q}=q_\omega^*}\tilde{r}(\hat{q},\omega) < 0$ therefore implies $\frac{\partial}{\partial \omega} q_\omega^* > 0$; hence, $\omega \mapsto q_\omega^*$ is strictly increasing, and so $q_1^* > q_0^*$.

Thus, all that remains is to show is that $\frac{\partial}{\partial \omega}\tilde{r}(\hat{q},\omega) < 0$ wherever $\tilde{r}(\hat{q},\omega)$ is

[40] This observation is the only part of the proposition that uses the condition that $\overline{v}_1 \geq \overline{v}_0$. In particular, the quantity ranking follows only from $\frac{\overline{v}_1}{\overline{v}_0}$ being strictly increasing on $(0,1]$.

zero. To that end, fix any $q^* \in (0,1)$ for the remainder of our analysis. Define now the continuously differentiable (by Lemma 8 and Lemma 9) functions

$$\begin{aligned} r^* : (0,1)\times[0,1] &\to \mathbb{R} \\ (q,\omega) &\mapsto \frac{1-q}{1-q^*} - 2 - \frac{2(1-q^*)}{\overline{v}_\omega(q)^2(1-q)\left[\Gamma^*_\omega(\overline{v}_\omega(q^*))-1\right]}\int_0^{\overline{v}_\omega(q)} p\left[1-\Gamma^*_\omega(p)\right]\mathrm{d}p \\ &= \frac{1-q}{1-q^*} - 2 - \frac{1-q^*}{\Gamma^*_\omega(\overline{v}_\omega(q^*))-1}\left[1 - \frac{1}{\overline{v}_\omega(q)^2(1-q)}\int_0^q \overline{v}_\omega^2\right] \end{aligned}$$

and

$$\begin{aligned} q_* : \{\omega\in[0,1]:\ \overline{v}_\omega(q^*) > \overline{p}^*_\omega\} &\to (0,1) \\ \omega &\mapsto \overline{v}_\omega^{-1}\left(\mathcal{P}_\omega(q^*)\right). \end{aligned}$$

Now, for any ω in the domain of q_*, the definition of $\mathcal{P}_\omega$ implies

$$0 = \int_{\overline{v}_\omega(q_*(\omega))}^{\overline{v}_\omega(q^*)}(1-\Gamma^*) = \int_0^{\overline{v}_\omega(q^*)}(1-\Gamma^*) - \int_0^{\overline{v}_\omega(q_*(\omega))}(1-\Gamma^*) = (1-q^*)\overline{v}_\omega(q^*) - \left[1-q_*(\omega)\right]\overline{v}_\omega(q_*(\omega)),$$

where the last identity follows from Lemma 8. It follows that every such ω has $\tilde{r}(q^*,\omega) = r^*(q_*(\omega),\omega)$. Therefore,

$$\frac{\partial}{\partial\omega}\tilde{r}(q^*,\omega) = \left[\frac{\partial}{\partial q}\bigg|_{q=q_*(\omega)} r^*(q,\omega)\right]q_*'(\omega) + \left[\frac{\partial}{\partial\omega}\bigg|_{q=q_*(\omega)} r^*(q,\omega)\right].$$

To show $\frac{\partial}{\partial\omega}\tilde{r}(q^*,\omega) < 0$, it thus suffices to show $\frac{\partial}{\partial q}\big|_{q=q_*(\omega)} r^*(q,\omega)$ and $\frac{\partial}{\partial\omega}\big|_{q=q_*(\omega)} r^*(q,\omega)$ are both strictly negative and $q_*'(\omega)$ is strictly positive. We pursue each of these three inequalities.

Toward signing $q_*'(\omega)$, observe the definition of $q_*(\cdot)$ and Lemma 8 imply

$$0 = \int_{\overline{v}_\omega(q_*(\omega))}^{\overline{v}_\omega(q^*)}(1-\Gamma^*) = (1-q^*)\overline{v}_\omega(q^*) - \left[1-q_*(\omega)\right]\overline{v}_\omega(q_*(\omega)),$$

which rearranges to

$$1 - q^* = [1 - q_*(\omega)] \frac{\overline{v}_\omega(q_*(\omega))}{\overline{v}_\omega(q^*)}.$$

Differentiating the above equation tells us

$$\begin{aligned} 0 &= [1 - q_*(\omega)] \frac{\partial}{\partial \omega}\Big|_{q=q_*(\omega)} \left[\frac{\overline{v}_\omega(q)}{\overline{v}_\omega(q^*)}\right] + q_*'(\omega) \frac{\partial}{\partial q}\Big|_{q=q_*(\omega)} \left[(1-q)\frac{\overline{v}_\omega(q)}{\overline{v}_\omega(q^*)}\right] \\ &< q_*'(\omega) \frac{\partial}{\partial q}\Big|_{q=q_*(\omega)} \left[(1-q)\frac{\overline{v}_\omega(q)}{\overline{v}_\omega(q^*)}\right] \\ &= q_*'(\omega) \frac{\overline{v}_\omega'(q_*(\omega))}{\overline{v}_\omega(q^*)} \left[1 - \Gamma^*(\overline{v}_\omega(q_*(\omega)))\right]. \end{aligned}$$

Hence, $q_*'(\omega)$ is strictly positive.

Now, to sign $\frac{\partial}{\partial \omega} r^*(q, \omega)$, it suffices to show

$$\frac{1 - \frac{1}{\overline{v}_\omega(q)^2 (1-q)} \int_0^q \overline{v}_\omega^2}{\Gamma_\omega^*(\overline{v}_\omega(q^*)) - 1}$$

has strictly positive partial derivative with respect to ω at $q = q_*(\omega)$. Because both the numerator and denominator are strictly positive there, it suffices to show the numerator is has positive partial derivative and denominator has negative partial derivative with respect to ω, at least one of them strictly so. First, the numerator's partial derivative is a strictly negative multiple of

$$\frac{\partial}{\partial \omega}\left[\frac{\int_0^q \overline{v}_\omega^2}{\overline{v}_\omega(q)^2}\right] = \frac{\partial}{\partial \omega}\int_0^q \left[\frac{\overline{v}_\omega(\hat{q})}{\overline{v}_\omega(q)}\right]^2 \, \mathrm{d}\hat{q} = 2\int_0^q \frac{\overline{v}_\omega(\hat{q})}{\overline{v}_\omega(q)} \frac{\partial}{\partial \omega}\left[\frac{\overline{v}_\omega(\hat{q})}{\overline{v}_\omega(q)}\right] \, \mathrm{d}\hat{q},$$

which is strictly negative. Second, the denominator's partial derivative is

$$\frac{\partial}{\partial \omega}\left[\Gamma_\omega^*(\overline{v}_\omega(q^*)) - 1\right] = \frac{\partial}{\partial \omega}\left[\frac{\overline{v}_\omega(q^*)}{\overline{v}_\omega'(q^*)}\right] = \frac{\partial}{\partial \omega}\left\{\frac{1}{\frac{\partial}{\partial q}\big|_{q=q^*} \log \overline{v}_\omega(q)}\right\},$$

which is nonpositive by log-supermodularity. So $\frac{\partial}{\partial \omega} r^*(q, \omega) < 0$.

Finally, we turn to signing $\frac{\partial}{\partial q} r^*(q, \omega)$. To that end, let $q := q_*(\omega)$, let $v := \overline{v}_\omega(q)$, and let $v' := \overline{v}_\omega'(q)$. Then, that $\Gamma_\omega^*(v) < 1$ rearranges to $(1-q)v' > v$.

Hence,

$$\begin{aligned}\frac{\partial}{\partial q}\left[\frac{\int_0^q \overline{v}_\omega^2}{\overline{v}_\omega(q)^2(1-q)}\right] &= \frac{v^2(1-q)v^2 - \left[2(1-q)vv' - v^2\right]\int_0^q \overline{v}_\omega^2}{v^4(1-q)^2} \\ &< \frac{v^2(1-q)v^2 - (2v^2 - v^2)\int_0^q \overline{v}_\omega^2}{v^4(1-q)^2} \\ &= \frac{1}{1-q}\left[1 - \frac{1}{(1-q)v^2}\int_0^q \overline{v}_\omega^2\right].\end{aligned}$$

Therefore, at such q we have

$$\begin{aligned}\frac{\partial}{\partial q} r^*(q,\omega) &= \frac{-1}{1-q^*} + \frac{1-q^*}{\Gamma_\omega^*(\overline{v}_\omega(q^*))-1}\cdot\frac{\partial}{\partial q}\left[\frac{\int_0^q \overline{v}_\omega^2}{\overline{v}_\omega(q)^2(1-q)}\right] \\ &< \frac{-1}{1-q^*} + \frac{1-q^*}{\Gamma_\omega^*(\overline{v}_\omega(q^*))-1}\left\{\frac{1}{1-q}\left[1 - \frac{1}{(1-q)v^2}\int_0^q \overline{v}_\omega^2\right]\right\} \\ &= \frac{-1}{1-q}\left[r^*(q,\omega)+2\right] \\ &= \frac{-2}{1-q} < 0.\end{aligned}$$

The quantity ranking follows. *Q.E.D.*

C.4. Proof of Proposition 5

First, we formulate our optimality condition more precisely. Say a tuple $(\Pi_1, \ldots, \Pi_N, q) \in [\Delta(\mathbb{R}_+)]^N \times [0,1]$ is *worst-feasible* if q is a worst equilibrium for the seller given that price distribution Π_n is used for each group $n \in \{1, \ldots, N\}$. Say a tuple $(\Pi_1^*, \ldots, \Pi_N^*, q^*) \in [\Delta(\mathbb{R}_+)]^N \times [0,1]$ is *limit-worst-feasible (LWF)* if it is a limit of a sequence of worst-feasible pairs. Let

$$R_N^* := \sup_{(\Pi_1,\ldots,\Pi_N,q) \text{ worst-feasible}} \sum_{n=1}^N \lambda_n R_{n,q}(\Pi_n)$$

denote the seller's optimal value. and say $(\Pi_1^*, \ldots, \Pi_N^*, q^*)$ is *optimal* if it is LWF with a witnessing sequence that has revenue converging to R_N^*.

Throughout this proof, for any notation used throughout our paper, and any $n \in \{1, \ldots, N\}$, let the same notation with a subscript n refer to the corresponding object for group n.

We begin our argument by noting that several results proven for our main model also apply directly to the setting with heterogeneity, with proofs applying verbatim or nearly verbatim. In particular, Lemma 1, Lemma 4, Lemma 5, Lemma 6, and the first sentence of Lemma 2 apply separately to each group; and the remainder of Lemma 2 (pertaining to the set of equilibrium quantities) applies with essentially the same proof. Moreover, following essentially identically the proof of Proposition 2 tells us $(\Pi_1^*, \ldots, \Pi_N^*, q^*)$ is optimal if and only if it solves program (P_N^*), and that such an optimum exists and yields strictly positive revenue. In addition, various claims proven in the proof of Theorem 1 adapt to the present setting with essentially identical proofs. First, Claim 1 and Claim 4 adapt immediately to each group separately. Next, the analogues of Claim 2 and Claim 5, in which we replace $q \mapsto D_q(\Pi)$ with the function $q \mapsto \sum_{n=1}^{N} \lambda_n D_{n,q}(\Pi_n)$ (and for Claim 5 replace the modified price distribution with a modified profile of N price distributions), adapt immediately. In what follows, whenever we reference any of these results, we mean to apply these analogues.

Now, fix some optimum $(\Pi_1^*, \ldots, \Pi_N^*, q^*)$ for program (P_N^*). It will be convenient to work with the "partial demand" functions $\mathcal{D}_n : [0,1] \to [0,1]$, for $n \in \{0, \ldots, N\}$, given by

$$\mathcal{D}_n(q) := \sum_{m=1}^{n} \lambda_m D_{m,q}(\Pi_m^*).$$

The following quantitative result tells us when mass can be swapped (and how much relative mass needs to be swapped) between two different locations (in "quantity space") for two different price distributions, while preserving the demand constraints and raising expected revenue.

Claim 6. *Suppose* $0 \le q_0 < q_1 \le q^*$ *and* $n_0, n_1 \in \{1, \ldots, N\}$ *have* $n_0 < n_1$. *Let* $v_0 := \overline{v}_{n_0}$ *and* $v_1 := \overline{v}_{n_1}$, *and take* $\gamma \in \mathbb{R}_+$. *Define* $\Delta D : [0,1] \to \mathbb{R}$ *and*

$\Delta R \in \mathbb{R}$ *by letting*

$$\begin{aligned}
\Delta D(q) &:= \gamma\left[D_{n_0,q}\left(v_0(q_0)\right) - D_{n_0,q}\left(v_0(q_1)\right)\right] + \left[D_{n_1,q}\left(v_1(q_1)\right) - D_{n_1,q}\left(v_1(q_0)\right)\right] \\
\Delta R &:= \gamma\left[R_{n_0,q^*}\left(v_0(q_0)\right) - R_{n_0,q^*}\left(v_0(q_1)\right)\right] + \left[R_{n_1,q^*}\left(v_1(q_1)\right) - R_{n_1,q^*}\left(v_1(q_0)\right)\right].
\end{aligned}$$

Then:

(i) Every $q \in [0, q_0]$ *has* $\Delta D(q) = 0$.

(ii) If $\gamma \geq \frac{v_0(q_1)}{v_1(q_1)} \cdot \frac{v_1(q_1)-v_1(q_0)}{v_0(q_1)-v_0(q_0)}$, *then every* $q \in [q_1, 1]$ *has* $\Delta D(q) \geq 0$.

(iii) If $q \in (q_0, q_1)$ *has* $\gamma \geq \frac{v_0(q)}{v_1(q)} \cdot \frac{v_1(q)-v_1(q_0)}{v_0(q)-v_0(q_0)}$, *then* $\Delta D(q) \geq 0$.

(iv) If $v_0(q_1) \leq p^M_{n_0}(q^*)$ *and* $\gamma \leq \frac{v_1(q_1)-v_1(q_0)}{v_0(q_1)-v_0(q_0)}$, *then* $\Delta R > 0$.

Proof. The first point follows trivially because each of the four demands in the definition of $\Delta D(q)$ is zero if $q \in [0, q_0]$.

Toward the second point, suppose γ satisfies the given inequality and $q \in [q_1, 1]$. That $D_{n_0,q}\left(v_0(q_0)\right) \geq D_{n_0,q}\left(v_0(q_1)\right)$ implies

$$\begin{aligned}
\Delta D(q) &\geq \tfrac{v_0(q_1)}{v_1(q_1)} \cdot \tfrac{v_1(q_1)-v_1(q_0)}{v_0(q_1)-v_0(q_0)}\left[D_{n_0,q}\left(v_0(q_0)\right) - D_{n_0,q}\left(v_0(q_1)\right)\right] \\
&\quad + \left[D_{n_1,q}\left(v_1(q_1)\right) - D_{n_1,q}\left(v_1(q_0)\right)\right] \\
&= \tfrac{v_0(q_1)}{v_1(q_1)} \cdot \tfrac{v_1(q_1)-v_1(q_0)}{v_0(q_1)-v_0(q_0)}\left\{\left[1 - \tfrac{v_0(q_0)}{v_0(q)}\right] - \left[1 - \tfrac{v_0(q_1)}{v_0(q)}\right]\right\} \\
&\quad + \left\{\left[1 - \tfrac{v_1(q_1)}{v_1(q)}\right] - \left[1 - \tfrac{v_1(q_0)}{v_1(q)}\right]\right\} \\
&= \tfrac{v_0(q_1)}{v_1(q_1)} \cdot \tfrac{v_1(q_1)-v_1(q_0)}{v_0(q_1)-v_0(q_0)}\left[-\tfrac{v_0(q_0)}{v_0(q)} + \tfrac{v_0(q_1)}{v_0(q)}\right] + \left[-\tfrac{v_1(q_1)}{v_1(q)} + \tfrac{v_1(q_0)}{v_1(q)}\right] \\
&= \tfrac{v_0(q_1)}{v_1(q_1)v_1(q)}\left[v_1(q_1) - v_1(q_0)\right]\left[\tfrac{v_1(q)}{v_0(q)} - \tfrac{v_1(q_1)}{v_0(q_1)}\right] \\
&\geq 0.
\end{aligned}$$

For the third point, suppose γ satisfies the given inequality for some given

$q \in (q_0, q_1)$. That $D_{n_0,q}(v_0(q_0)) \geq D_{n_0,q}(v_0(q_1))$ implies

$$\begin{aligned}
\Delta D(q) &\geq \tfrac{v_0(q)}{v_1(q)} \cdot \tfrac{v_1(q)-v_1(q_0)}{v_0(q)-v_0(q_0)} \left[D_{n_0,q}(v_0(q_0)) - D_{n_0,q}(v_0(q_1))\right] \\
&\quad + \left[D_{n_1,q}(v_1(q_1)) - D_{n_1,q}(v_1(q_0))\right] \\
&= \tfrac{v_0(q)}{v_1(q)} \cdot \tfrac{v_1(q)-v_1(q_0)}{v_0(q)-v_0(q_0)} D_{n_0,q}(v_0(q_0)) - D_{n_1,q}(v_1(q_0)) \\
&= \tfrac{v_0(q)}{v_1(q)} \cdot \tfrac{v_1(q)-v_1(q_0)}{v_0(q)-v_0(q_0)} \left[1 - \tfrac{v_0(q_0)}{v_0(q)}\right] - \left[1 - \tfrac{v_1(q_0)}{v_1(q)}\right] \\
&= 0.
\end{aligned}$$

Finally, for the fourth point, suppose $v_0(q_1) \leq p^M_{n_0}(q^*)$ and γ satisfies the given inequality. That $v_0(q_1) \leq p^M_{n_0}(q^*)$ implies that $q_1 < q^*$ and (given Assumption 3) that $R_{n_0,q^*}(v_0(q_0)) < R_{n_0,q^*}(v_0(q_1))$. Therefore,

$$\begin{aligned}
\Delta R &\geq \tfrac{v_1(q_1)-v_1(q_0)}{v_0(q_1)-v_0(q_0)} \left[R_{n_0,q^*}(v_0(q_0)) - R_{n_0,q^*}(v_0(q_1))\right] \\
&\quad + \left[R_{n_1,q^*}(v_1(q_1)) - R_{n_1,q^*}(v_1(q_0))\right] \\
&= \tfrac{v_1(q_1)-v_1(q_0)}{v_0(q_1)-v_0(q_0)} \left\{v_0(q_0)\left[1 - \tfrac{v_0(q_0)}{v_0(q^*)}\right] - v_0(q_1)\left[1 - \tfrac{v_0(q_1)}{v_0(q^*)}\right]\right\} \\
&\quad + \left\{v_1(q_1)\left[1 - \tfrac{v_1(q_1)}{v_1(q^*)}\right] - v_1(q_0)\left[1 - \tfrac{v_1(q_0)}{v_1(q^*)}\right]\right\} \\
&= \tfrac{v_1(q_1)-v_1(q_0)}{v_0(q_1)-v_0(q_0)} \left[v_0(q_0) - v_0(q_1)\right] + \left[v_1(q_1) - v_1(q_0)\right] \\
&\quad + \tfrac{v_1(q_1)-v_1(q_0)}{v_0(q_1)-v_0(q_0)} \left[-\tfrac{v_0(q_0)^2}{v_0(q^*)} + \tfrac{v_0(q_1)^2}{v_0(q^*)}\right] + \left[-\tfrac{v_1(q_1)^2}{v_1(q^*)} + \tfrac{v_1(q_0)^2}{v_1(q^*)}\right] \\
&= \tfrac{v_1(q_1)-v_1(q_0)}{v_0(q_1)-v_0(q_0)} \left[-\tfrac{v_0(q_0)^2}{v_0(q^*)} + \tfrac{v_0(q_1)^2}{v_0(q^*)}\right] + \left[-\tfrac{v_1(q_1)^2}{v_1(q^*)} + \tfrac{v_1(q_0)^2}{v_1(q^*)}\right] \\
&= \left[v_1(q_1) - v_1(q_0)\right] \left[\tfrac{v_0(q_0)+v_0(q_1)}{v_0(q^*)} - \tfrac{v_1(q_1)+v_1(q_0)}{v_1(q^*)}\right] \\
&= \tfrac{v_1(q_1)-v_1(q_0)}{v_1(q^*)} \left\{v_0(q_0)\left[\tfrac{v_1(q^*)}{v_0(q^*)} - \tfrac{v_1(q_0)}{v_0(q_0)}\right] + v_0(q_1)\left[\tfrac{v_1(q^*)}{v_0(q^*)} - \tfrac{v_1(q_1)}{v_0(q_1)}\right]\right\} \\
&> 0,
\end{aligned}$$

where strictness in the last inequality follows from $0 < q_1 < q^*$. *Q.E.D.*

The following claim establishes that gaps (in "quantity space") in the price distribution are always bookended by atoms.

Claim 7. *Suppose* $0 \leq q_0 < q_1 < q^*$ *and* $[q_0, q_1] \cap \operatorname{supp} \overline{\Pi} = \{q_0, q_1\}$, *where* $\overline{\Pi} := \sum_{n=1}^{N} \lambda_n \Pi^*_n \circ \overline{v}_n$. *Then,* $\overline{\Pi}$ *has mass points at both* q_0 *and* q_1.

Proof. We proceed in two cases. First, consider the case in which $\mathcal{D}_N(q_0) = q_0$ [resp. $\mathcal{D}_N(q_1) = q_1$]. Note that Lemma 5 tells us the function $q \mapsto \mathcal{D}_N(q) - q$ is strictly concave on (q_0, q_1), hence (because it is nonnegative) strictly positive. Therefore, in this case, combining Claim 2 and Lemma 6 (exactly as in the proof of Claim 3) tells us $\overline{\Pi}$ has a mass point at q_0 [resp. q_1].

Second, consider the case in which $\mathcal{D}_N(q_0) > q_0$ [resp. $\mathcal{D}_N(q_1) > q_1$]. Assume for a contradiction that $\overline{\Pi}$ does not have a mass point at q_0 [resp. q_1]—or equivalently, that $\overline{\Pi}_n := \Pi_n^* \circ \overline{v}_n$ does not have a mass point there for any $n \in \{1, \ldots, N\}$. Lemma 1 then implies $\mathcal{D}_N(q) > q$ for every q in some open interval Q around q_0 [resp. q_1]. Now, take some $n \in \{1, \ldots, N\}$ such that q_0 [resp. q_1] is in $\operatorname{supp} \overline{\Pi}_n$, which exists because it is in $\operatorname{supp} \overline{\Pi}$. Because it has a support point somewhere in Q where it does not have a mass point, $\overline{\Pi}_n$ must have multiple support points in Q. But then the mass in this interval can be replaced with its expectation. Doing so will preserve $\mathcal{D}_N(q)$ for $q \in (0, q^*) \setminus Q$, and will strictly improve the objective by Lemma 4. Moreover, Claim 5 tells us an appropriate proper weighted average of $\overline{\Pi}_n$ and the modified distribution also satisfies $\mathcal{D}_N(q)$ for $q \in Q$, in contradiction to the optimality of $(\overline{\Pi}_n)_{n=1}^N$ in program ($\overline{\mathrm{P}}_N^*$). *Q.E.D.*

The following claim, together with (an appropriate generalization of) Theorem 1, is the heart of the proposition.

Claim 8. *Every* $n \in \{1, \ldots, N\}$ *has* $\Pi_n^*(p_n^M(q^*)) = 1$ *and, if* $n < N$*, has*

$$\max \operatorname{supp}(\Pi_n^* \circ \overline{v}_n) \leq \min \operatorname{supp}(\Pi_{n+1}^* \circ \overline{v}_{n+1}).$$

Proof. First, observe $\Pi_n^*(p_n^M(q^*)) = 1$ for every $n \in \{1, \ldots, N\}$. Indeed, if not, then applying Claim 1 for some group n with $\Pi_n^*(p_n^M(q^*)) < 1$ would violate optimality.

Letting $\overline{\Pi}_n := \Pi_n^* \circ \overline{v}_n \in \Delta[0, q^*]$ for each $n \in \{1, \ldots, N\}$, we now know that $(\overline{\Pi}_n)_{n=1}^N$ is an optimal solution to the program

$$\max_{(\hat{\Pi}_n)_{n\in(\Delta[0,q^*])^N}} \sum_{n=1}^{N} \lambda_n R_{n,q^*}(\hat{\Pi}_n \circ \underline{q}_n) \tag{$\overline{\mathrm{P}}_N^*$}$$

$$\text{subject to} \quad \sum_{n=1}^{N} \lambda_n D_{n,\hat{q}}(\hat{\Pi}_n \circ \underline{q}_n) \geq \hat{q} \quad \forall \hat{q} \in (0, q^*).$$

In what follows, let us say (n_0, n_1, q_1, q_0) is a *mismatch* if $n_0, n_1 \in \{1, \ldots, N\}$ have $n_0 < n_1$, and $q_1 \in \operatorname{supp} \overline{\Pi}_{n_0}$ and $q_0 \in \operatorname{supp} \overline{\Pi}_{n_1}$ have $q_0 < q_1$. It remains to show that no mismatch exists.

Now, let us argue no $\hat{q} \in (0, q^*) \cap \left(\operatorname{supp} \overline{\Pi}_{n_0}\right) \cap \left(\operatorname{supp} \overline{\Pi}_{n_1}\right)$ is such that $\Pi^*_{n_0}$ has mass at $\hat{q}^{++}$ or $\Pi^*_{n_1}$ has mass at $\hat{q}^{--}$. Assume for a contradiction that such $\hat{q}$ does exist; we will show this assumption contradicts optimality in ($\overline{\mathrm{P}}_N^*$). To that end, note that the externality ranking implies $\frac{v_1(\hat{q})}{v_0(\hat{q})} > 1$, so that some $\gamma \in \left(\frac{v_0(\hat{q})v_1'(\hat{q})}{v_1(\hat{q})v_0'(\hat{q})}, \frac{v_1'(\hat{q})}{v_0'(\hat{q})}\right)$ exists. Because v_0, v_1 are continuously differentiable, strictly increasing, and strictly positive on $(0, q^*)$, some neighborhood of $\hat{q}$ must exist in $(0, q^*)$ such that any $q_0 < q_1$ in the neighborhood have

$$\frac{v_0(q_1)}{v_1(q_1)} \cdot \frac{v_1(q_1)-v_1(q_0)}{v_0(q_1)-v_0(q_0)} < \gamma < \frac{v_1(q_1)-v_1(q_0)}{v_0(q_1)-v_0(q_0)}.$$

By definition of $\hat{q}$, there exist some probability measures $\hat{\Pi}_0, \hat{\Pi}_1$ on this neighborhood such that some strictly positive multiple of $\hat{\Pi}_0$ is a submeasure of $\overline{\Pi}_{n_0}$, some strictly positive multiple of $\hat{\Pi}_1$ is a submeasure of $\overline{\Pi}_{n_1}$, and $\max \operatorname{supp} \hat{\Pi}_1 < \min \operatorname{supp} \hat{\Pi}_0$.[41] So let $\varepsilon > 0$ be small enough to ensure $\frac{\varepsilon\gamma}{\lambda_{n_0}}\hat{\Pi}_0$ is a submeasure of $\overline{\Pi}_{n_0}$ and $\frac{\varepsilon}{\lambda_{n_1}}\hat{\Pi}_1$ is a submeasure of $\overline{\Pi}_{n_1}$. We can then replace $\overline{\Pi}_{n_0}$ with $\overline{\Pi}_{n_0} + \frac{\varepsilon\gamma}{\lambda_{n_0}}\left(\hat{\Pi}_1 - \hat{\Pi}_0\right)$ and replace $\overline{\Pi}_{n_1}$ with $\overline{\Pi}_{n_1} + \frac{\varepsilon}{\lambda_{n_1}}\left(\hat{\Pi}_0 - \hat{\Pi}_1\right)$; Claim 6 ensures that this modified tuple of distributions remains feasible in ($\overline{\mathrm{P}}_N^*$) and strictly improves the objective—the desired contradiction.

Next, we argue that there cannot exist a mismatch (n_0, n_1, q_0, q_1) with

[41] Recall, a submeasure of a measure is any measure that assigns a weakly lower mass to any event. In the language of cumulative distribution functions, $\tilde{\Gamma}$ is a submeasure of Γ if $\Gamma - \tilde{\Gamma}$ is weakly increasing.

$[q_0, q_1] \cap \operatorname{supp} \overline{\Pi} = \{q_0, q_1\}$, where $\overline{\Pi}$ is as defined in Claim 7. Indeed, assume otherwise for a contradiction, with the witnessing n_0, n_1 chosen to maximize $n_1 - n_0 > 0$. Let us now argue $\overline{\Pi}_{n_1}$ has a mass point at q_0, and $\overline{\Pi}_{n_0}$ has a mass point at q_1. To that end, note Claim 7 tells us $\overline{\Pi}_{\tilde{n}_1}$ [resp. $\overline{\Pi}_{\tilde{n}_0}$] has a mass point at q_0 [resp. q_1] for some $\tilde{n}_1 \in \{1, \ldots, N\}$ [resp. $\tilde{n}_0 \in \{1, \ldots, N\}$], and maximality of $n_1 - n_0 > 0$ tells us $\tilde{n}_1 \leq n_1$ [resp. $\tilde{n}_0 \geq n_0$]. If $n_1 = \tilde{n}_1$ [resp. $n_0 = \tilde{n}_0$], then $\overline{\Pi}_{n_1}$ [resp. $\overline{\Pi}_{n_0}$] has a mass point at q_0 [resp. q_1] by definition. If $n_1 < \tilde{n}_1$ [resp. $n_0 > \tilde{n}_0$], then $\overline{\Pi}_{n_1}$ [resp. $\overline{\Pi}_{n_0}$] has a mass point at q_0 [resp. q_1] by the previous paragraph—for otherwise $\overline{\Pi}_{n_1}$ [resp. $\overline{\Pi}_{n_0}$] would have mass immediately to the left [resp. right] of q_0 [resp. q_1]. Thus, $\overline{\Pi}_{n_1}$ [resp. $\overline{\Pi}_{n_0}$] has a mass point at q_0 [resp. q_1]. Fixing any $\gamma \in \left(\frac{v_0(q_1)}{v_1(q_1)} \cdot \frac{v_1(q_1) - v_1(q_0)}{v_0(q_1) - v_0(q_0)}, \frac{v_1(q_1) - v_1(q_0)}{v_0(q_1) - v_0(q_0)} \right)$, we can then find some $\varepsilon > 0$ small enough that $\overline{\Pi}_{n_0}$ puts mass at least $\frac{\varepsilon \gamma}{\lambda_{n_0}}$ on q_1 and $\overline{\Pi}_{n_1}$ puts mass at least $\frac{\varepsilon}{\lambda_{n_1}}$ on q_0. We can then replace $\overline{\Pi}_{n_0}$ with $\overline{\Pi}_{n_0} + \frac{\varepsilon \gamma}{\lambda_{n_0}} \left(\mathbf{1}_{[q_0, \infty)} - \mathbf{1}_{[q_1, \infty)} \right)$ and replace $\overline{\Pi}_{n_1}$ with $\overline{\Pi}_{n_1} + \frac{\varepsilon}{\lambda_{n_1}} \left(\mathbf{1}_{[q_1, \infty)} - \mathbf{1}_{[q_0, \infty)} \right)$. Claim 6 ensures that this modified tuple of distributions strictly improves the objective in ($\overline{\mathrm{P}}_N^*$), and preserves the constraint that $\sum_{n=1}^N \lambda_n D_{n, \hat{q}}(\hat{\Pi}_n \circ \underline{q}_n) \geq \hat{q} \;\; \forall \hat{q} \in (0, q^*)$ for any $\hat{q} \in (0, q^*) \backslash (q_0, q_1)$. Claim 5 then tells us this modification is feasible in ($\overline{\mathrm{P}}_N^*$)—contradicting optimality—if ε is sufficiently small.

Now, let us show no mismatch (n_0, n_1, q_0, q_1) exists with $q_0, q_1 > 0$. Indeed, assume otherwise for a contradiction. For any n_0, n_1 and any $\tilde{q} \in (0, 1)$, let

$$Q(n_0, n_1, \tilde{q}) := \left\{ (q_0, q_1) \in [\tilde{q}, 1]^2 : \; (n_0, n_1, q_0, q_1) \text{ is a mismatch} \right\}.$$

The contradiction hypothesis tells us that that sufficiently small $\tilde{q} \in (0, 1)$ has

$$Q(\tilde{q}) := \bigcup_{n_0, n_1 \in \{1, \ldots, N\}: \; n_0 < n_1} Q(n_0, n_1, \tilde{q}) \neq \emptyset.$$

Fixing such a $\tilde{q}$, let $(q_0^k, q_1^k)_{k=1}^\infty$ be some sequence from $Q(\tilde{q})$ such that $q_1^k - q_0^k$ converges as $k \to \infty$ to $\inf \{q_1 - q_0 : \; (q_0, q_1) \in Q(\tilde{q})\}$. Dropping to a subsequence, we may assume that the sequences $(q_0^k)_{k=1}^\infty$ and $(q_1^k)_{k=1}^\infty$ are both monotone (hence convergent), and that some $n_0, n_1 \in \{1, \ldots, N\}$ with $n_0 < n_1$ have

$\left\{(q_0^k, q_1^k)\right\}_{k=1}^{\infty} \subseteq Q(n_0, n_1, \tilde{q})$. We now derive a contradiction in each of two cases. First, if $\lim_{k\to\infty}\left(q_1^k - q_0^k\right) = 0$, then n_0 and n_1, together with $\hat{q} := \lim_{k\to\infty} q_0 = \lim_{k\to\infty} q_1$, contradicts the claim of the paragraph before the previous one. Second, if $\lim_{k\to\infty}\left(q_1^k - q_0^k\right) > 0$, then $(q_0, q_1) := \lim_{k\to\infty}(q_0^k, q_1^k)$ belongs to $Q(n_0, n_1, \tilde{q})$ and satisfies $q_1 - q_0 = \min\left\{\hat{q}_1 - \hat{q}_0 : \ (\hat{q}_0, \hat{q}_1) \in Q(\tilde{q})\right\}$. But this minimality tells us no $n \in \{1, \ldots, N\}$ can have any $q \in (q_0, q_1) \cap \operatorname{supp} \overline{\Pi}_n$: if $n < n_1$ we would have $(q_0, q) \in Q(n, n_1, \tilde{q})$, and if $n > n_0$ we would have $(q, q_1) \in Q(n_0, n, \tilde{q})$. But then (n_0, n_1, q_0, q_1) contradicts the claim of the previous paragraph.

All that remains is to show we cannot have a mismatch of the form $(n_0, n_1, 0, q_1)$. Assume otherwise, for a contradiction. If some $q_0 \in (0, q_1) \cap \operatorname{supp} \overline{\Pi}_{n_1}$ exists, then (n_0, n_1, q_0, q_1) is a mismatch with $q_0, q_1 > 0$—a contradiction. So we can now focus on the case that no such q_0 exists, implying $\overline{\Pi}_{n_1}$ has a mass point at zero. Therefore, letting $\varepsilon := \min\left\{\lambda_{n_0}, \lambda_{n_1}\overline{\Pi}_{n_1}(0)\right\} > 0$, we can then replace $\overline{\Pi}_{n_0}$ with $\overline{\Pi}_{n_0} + \frac{\varepsilon}{\lambda_{n_0}}\left(\mathbf{1}_{[0,\infty)} - \overline{\Pi}_{n_0}\right)$ and replace $\overline{\Pi}_{n_1}$ with $\overline{\Pi}_{n_1} + \frac{\varepsilon}{\lambda_{n_1}}\left(\overline{\Pi}_{n_0} - \mathbf{1}_{[0,\infty)}\right)$; Claim 6 ensures that this modified tuple of distributions remains feasible in ($\overline{\mathrm{P}}_N^*$) and strictly improves the objective—the desired contradiction. *Q.E.D.*

Claim 8 in particular tells us for each $n \in \{1, \ldots, N\}$ that we can view

$$\overline{\Pi}_n := \Pi_n^* \circ \overline{v}_n$$

as an element of $\Delta[0, q^*]$. Let $q_n^0 := \min \operatorname{supp} \overline{\Pi}_n$ and $q_n^1 := \max \operatorname{supp} \overline{\Pi}_n$, and observe that $p_n^* := \max \operatorname{supp} \Pi_n^* = \overline{v}_n(q_n^1)$ by construction.

The next claim adapts arguments from the greediness proof in Theorem 1 and its Corollary 1 to the present setting.

Claim 9. *Every $n \in \{1, \ldots, N\}$ has $\operatorname{supp} \Pi_n^* = \left[\underline{q}_n(q_n^0), p_n^*\right]$ and*

$$\mathcal{D}_N(q) = \mathcal{D}_n(q) = q, \ \forall q \in (q_n^0, q_n^1).$$

Moreover, Π_N^* has a mass point at p_N^*.

Proof. First observe that $\mathcal{D}_n$ and $\mathcal{D}_N$ agree on $[0, q_n^1]$ by Claim 8. The rest of the claim holds vacuously if $q_n^0 = q_n^1$, so focus now on the nontrivial case in which $q_n^0 < q_n^1$.

To show the result, we follow the proof of Theorem 1. We have already noted how Claim 1, Claim 2, Claim 4, and Claim 5 adapt. Moreover, an analogue of Claim 3 goes through—with the set $\{q \in (0, \underline{q}(p^*)) : D_q(\Pi) > q\}$ being replaced by $\{q \in (q_n^0, q_n^1) : \mathcal{D}_N(q) > q\}$ for some group n—given that Claim 8 says $\overline{\Pi}_m$ has no mass on (q_n^0, q_n^1) for $m \neq n$.

Applying the above five (modified) claims analogously to in the proof of Theorem 1, we learn that $(\Pi_1^*, \ldots, \Pi_N^*, q^*)$ could be modified to maintain feasibility in program (P_N^*) and strictly improve the objective if $\mathcal{D}_N(q) > q$ held for some $q \in (q_n^0, q_n^1)$. Thus, optimality tells us $\mathcal{D}_N(q) = q$ for every $q \in (q_n^0, q_n^1)$.

Moreover, adapting the proof of Lemma 7, we learn that $\{\overline{\Pi}_m\}_{m=1}^N$ cannot all be constant over any subinterval of (q_n^0, q_n^1). Thus (given Claim 8) $\overline{\Pi}_n$ is strictly increasing on this interval, meaning its support is equal to $[q_n^0, q_n^1]$ exactly.

Finally, following the argument for a mass point at the top of the support in the proof of Theorem 1 tells at least one of $\{\overline{\Pi}_n\}_{n=1}^N$ has a mass point at $\max\operatorname{supp}\left[\sum_{n=1,\ldots,N} \overline{\Pi}_n\right]$, which is equal to q_N^1 by Claim 8. But again by Claim 8, either $q_n^1 < q_N^1$ for every $n \in \{1, \ldots, N-1\}$ or $\overline{\Pi}_N$ is degenerate on q_N^1; in either case, $\overline{\Pi}_N$ has a mass point at q_N^1. Thus, Π_N^* has a mass point at p_N^*. *Q.E.D.*

With these two claims in hand, we are now prepared to prove the proposition. Toward the support ranking, observe any $n \in \{1, \ldots, N-1\}$ has

$$\min\operatorname{supp}\Pi_{n+1}^* = \overline{v}_{n+1}(q_{n+1}^0) \geq \overline{v}_n(q_{n+1}^0) \geq \overline{v}_n(q_n^1) = \max\operatorname{supp}\Pi_n^*, \quad (9)$$

where the first inequality follows from $\overline{v}_{n+1}$ having stronger externalities than $\overline{v}_n$, and the second inequality follows from Claim 8. Toward the strict ranking, it suffices to see we cannot have $q_{n+1}^0 = 0 < q_{n+1}^1$. Indeed, $\min\operatorname{supp}\Pi_{n+1}^* > \max\operatorname{supp}\Pi_n^*$ would then readily follow: If $q_{n+1}^0 > 0$, then $\overline{v}_{n+1}$ having stronger externalities than $\overline{v}_n$ would mean that in fact the first inequality in (9) holds strictly; and if $q_{n+1}^1 = 0$, then Claim 8 would tell us $p_n^* = p_{n+1}^* = 0$. So let us assume $q_{n+1}^0 = 0 < q_{n+1}^1$ for a contradiction. Claim 8 then tells us Π_n^* is degenerate on 0, so that every $q \in [0,1]$ has $\mathcal{D}_N(q) \geq \mathcal{D}_N(0) \geq \lambda_n > 0$, whereas Claim 9 tells us sufficiently small $q \in (0, \lambda_n)$ has $\mathcal{D}_N(q) = q$—a contradiction.

It remains to show that $(\Pi_n^*)_{n=1}^N$ are residual greedy up to $(p_n^*)_{n=1}^N$. In light of Claim 9, we need only show for each $n \in \{1, \dots, N\}$ that

$$q_{n-1} := \max \{q \in [0,1] : \ \mathcal{D}_{n-1}(q) = q\}$$

is well-defined, that $q_n^0 = q_n^1$ if $q_n^1 \leq q_{n-1}$, and that $q_n^0 = q_{n-1}$ if $q_n^1 > q_{n-1}$. All three claims are immediate if $n = 1$, because $q_0 = 0$ by definition, $0 \leq q_1^0 \leq q_1^1$, and feasibility in (P_N^*) requires (given that $q_1^0 = \min_{m=1,\dots,N} q_m^0$ by Claim 8) that $q_1^0 = 0$. So let us focus on the nontrivial case in which $n > 1$. Observe now that $q \mapsto \mathcal{D}_{n-1}(q) - q$ is nonnegative at q_{n-1}^1 (given feasibility in (P_N^*)), nonpositive at 1, and continuous (by Lemma 1) and strictly concave (by Lemma 5) on $\left[q_{n-1}^1, 1\right]$. The function either crosses zero from above exactly once in $(q_{n-1}^1, 1]$, or it is strictly decreasing on $[q_{n-1}^1, 1]$. In either case, q_{n-1} is well-defined, and $\mapsto \mathcal{D}_{n-1}(q) - q$ is strictly positive on (q_{n-1}^1, q_{n-1}) and strictly negative on $(q_{n-1}, 1]$.

We now need to show (for $n > 1$) that $q_n^0 = q_n^1$ if $q_n^1 \leq q_{n-1}$, and that $q_n^0 = q_{n-1}$ if $q_n^1 > q_{n-1}$. Toward the first assertion, suppose $q_n^1 \leq q_{n-1}$. Then any $q \in (q_{n-1}^1, q_n^1)$ has $\mathcal{D}_N(q) \geq \mathcal{D}_{n-1}(q) > q$. Meanwhile, Claim 9 says any $q \in (q_n^0, q_n^1)$ has $\mathcal{D}_n(q) = q$. Hence, (q_n^0, q_n^1) and (q_{n-1}^1, q_n^1) are disjoint, implying $q_n^0 = q_n^1$ as desired. Finally, toward the second assertion, suppose $q_n^1 > q_{n-1}$. We want to show $q_n^0 = q_{n-1}$. If any $q \in (q_{n-1}, q_n^0)$ existed, it would (given Claim 8) have $\mathcal{D}_N(q) = \mathcal{D}_{n-1}(q) < q$, in violation of feasibility in (P_N^*). And if any $q \in (q_n^0, q_{n-1})$ existed, it would have $\mathcal{D}_N(q) \geq \mathcal{D}_{n-1}(q) > q$, in contradiction to Claim 9. Thus, no q lies strictly between q_n^0 and q_{n-1}, meaning they coincide. The proposition follows. *Q.E.D.*