Reputational Delegation[*]

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Abstract

I study how a principal should delegate to an agent with career concerns. The agent and principal are assumed to have aligned intrinsic incentives. However, the market observes the chosen action, and rewards the agent more, the higher it perceives his (private) type to be. This "reputational bias" has many similarities to the classic "material bias" studied in the communication literature, for instance, both can induce the same cheap talk equilibrium sets. However, I show that it is *always* optimal to impose a floor on the set of available actions in reputational delegation. This is in stark contrast to delegation to an agent with a material bias, where it is *never* optimal to restrict the agent's flexibility to take low actions. I specialize to the exponential family of distributions to show that offering flexibility to high types (i.e. full separation) is optimal. This result uses a recursive approach novel to communication problems.

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1. Introduction

Delegation is useful when the individual with decision rights is not the individual with the most relevant information. For example, school administrators delegate grading to teachers who better know the true ability of each student, private equity investors cede investment decisions to the fund's general partners, and law firm partners allow associates to decide how many billable hours to put into each case. The key tension is that the agent – the individual with more information – may be biased relative to the principal – the individual with decision rights.

A focal case is when the agent is biased toward higher decisions relative to the principal, e.g. a teacher wants to give students higher grades than are warranted by their performance, a venture capitalist wants to invest more than necessary in each entrepreneur, and an associate wants to put excess hours into each case. There are many reasons that such biases may arise. For example, the teacher may like their students and want them to do well, the venture capitalist may face limited downside risk relative to the end investors, and the associate's bonus may be positively tied to their hours worked. These agents have a *material bias*: an exogenous preference for higher actions than the principal would take with the same information. The large literature on delegation initiated by Holmstrom (1984) studies how a principal should delegate to an agent with a material bias.

Alternatively, the agents could have a *reputational bias* towards choosing higher actions. This arises when situations that warrant a higher action are associated with those in which the agent has higher ability. For example, better teachers educate their students to perform better and therefore tend to give higher grades, better venture capitalists find more promising entrepreneurs who warrant larger investments, and more dedicated associates put more work into each case. In each of these examples the agent has an incentive to choose an even higher action in order to signal higher ability. In contrast to a material bias, a reputational bias is endogenous.

There is often a subtle distinction between when an agent's incentives induce a reputational bias and when they induce a material bias. As a result, many economic interactions that motivate classic delegation studies could alternatively involve a reputationally biased agent. For example, workers may be biased towards working excessive hours in order to earn higher bonuses. Law firm associates receive a bonus if they work more than some predetermined amount of hours within a year, while investment banking analysts receive a bonus that depends on their manager's subjective view of their performance. In the law firm, bonuses induce a material bias towards working more hours: there is an exogenous relationship between hours worked and bonuses. Conversely, in the investment bank, bonuses induce a reputational bias towards more hours: an analyst's incentives depend endogenously on how many hours less-dedicated analysts work, and his desire to separate from them in the eyes of the manager. This motivates the main questions of this paper: how should a principal delegate to an agent with a reputational bias, and how does this differ from that for an agent with a material bias?

My model is the same as the standard delegation model, except for the agent's preferences. It features a single principal who wants to match his chosen action to an agent's private type, both of which are positive real numbers. The agent wants to match the action to their type, that is, their material component is aligned with the principal. However, the agent also benefits from giving the market – which only observes the action he chooses – the impression that he is a higher type. The principal delegates a subset of actions to the agent. I focus on the direct mechanism approach, in which the principal optimizes over incentive compatible *allocations*, i.e. mappings from agent types to actions.[1]

The main takeaway from the material delegation literature is that the principal should "cap against the agent's bias" (see Holmstrom (1984), Alonso and Matouschek (2008), Amador and Bagwell (2013)). This has two dimensions: (i) the agent is restricted from taking high actions, and (ii) the principal should never restrict the agent's flexibility to take low actions.[2] In contrast, Theorem 1 shows that in reputational delegation the principal always wants to impose a floor. That is, the principal pools some lower set of agent types by restricting them to take an inefficiently high action.

Theorem 1 leaves open whether a floor, i.e. pooling low types, is beneficial, or if it is simply pooling that is beneficial or necessary, as in cheap talk (Crawford and Sobel (1982)). Answering this question requires more detail about the optimal delegation set, for which I specialize to the exponential family of type distributions. Theorem 2 shows that the optimal allocation is eventually separating, i.e. above some type, the agent perfectly reveals his type with his chosen action. I also show that for exponential distributions close to the uniform limit, the optimal delegation set involves a floor and then full separation (flexibility) above some action. Methods for solving the material delegation problem, e.g. those in Kleiner et al. (2021) and Amador and Bagwell (2013), do not work in the reputational delegation model. Instead, I solve the principal's problem using recursive methods that

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[1] One difference from the material delegation framework is that following delegation, the agent and the market play a Bayesian game with potentially multiple equilibria. In Section 5 I show that under a regularity assumption on the distribution of types, there is a unique equilibrium that survives the D1 refinement, and that this equilibrium uses all delegated actions. In this sense, the principal can uniquely implement any incentive compatible allocation by delegating the set of on-path actions.

[2] Even when interval delegation is not optimal in the material delegation model, it is always optimal to give the lowest type his ideal action and also make all lower actions available.

are (to my knowledge) novel in communication models.

To interpret the main results, note that in the exponential model, the optimal delegation set is of the form $\{a_0\} \cup [\underline{a}, \infty)$, where $0 < a_0 < \underline{a}$. In general, setting a floor means the principal restricts flexibility at the bottom to some isolated action a_0. In contrast, under material delegation with some basic regularity assumptions, it is optimal to fully delegate in the exponential model.[3] In the three economic examples above, setting a floor can be interpreted as: pooling all low performing students on an F grade, setting some minimum investment for contracted entrepreneurs, or forcing associates to put at least a fixed amount of hours into each case.

While there are many alternative explanations for these delegation policies, the intuition in the current model comes from the optimal floor reducing distortionary signaling incentives for high ability agents. That is, setting an optimal floor reduces incentives to teach to the test in order to signal a quality education, to over-invest in order to signal prescience, or to put in inefficient hours in order to signal diligence. The mechanism behind this derives from the endogeneity of the reputational bias, i.e., the agent's incentive to take a higher action depends on the allocation. This opens up a channel for the principal to sacrifice efficiency for a set of agent types in order to decrease the effective bias on some other group of types. Indeed, this is the intuition for Theorem 1: the principal sacrifices loss on low types in order to improve the allocation for higher types. For example, relative to full delegation in the exponential model, constraining the agent to take actions in $\{a_0\} \cup [\underline{a}, \infty)$ may increase loss on types who select the floor a_0, but achieves less distortion for types who choose the higher actions $[\underline{a}, \infty)$ in return.

The methodology behind both main results exploits the following recursive structure. Consider an arbitrary incentive compatible allocation x with a threshold at some agent type t, i.e., all types below t get different actions than all types above t. I refer to a *continuing allocation* at t as an allocation of actions to types above t which would maintain incentive compatibility when juxtaposed with x below t. This set depends on x only through the loss experienced by type t. Correspondingly, I refer to $V(t, u)$ as the minimized continuation loss for the principal among the set of continuing allocations at t given that type t experiences loss u from the allocation below.

If x is optimal, naturally, it must minimize loss for the principal among all other IC allocations with a threshold t. With the definition of V, this means that x must minimize the loss below t plus the minimized continuation loss after t. For a low enough threshold

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[3] There is no optimal cap in the standard quadratic loss material delegation model with an exponential type distribution. Indeed, this is generically true for many unbounded type distributions. In general an interval of actions near 0 is in every optimal delegation set.

type t, the difference in loss below type t across allocations is relatively small. Thus, the optimal choice for the allocation below type t comes down to which one induces the lowest continuation loss above t. The key to answering this question is Proposition 1 – termed *the alignment principle* – which says that $V(t, u)$ has a strictly positive derivative in u. That is, increasing the agent's loss at type t increases the principal's minimized continuation loss above type t. Thus, given the observation above, for a small enough threshold t, the optimal allocation must minimize the loss at t.

I then show that an allocation that pools some set of types $[0, t]$, i.e. implements a floor, achieves this goal of minimizing loss for a small threshold type t. The intuition is as follows. In order for an allocation to separate a higher type from a lower type, the higher type must be given a sufficiently high action to negatively compensate for the associated increased reputation and deter the lower type from deviating. As a result, finely separating types causes distortion (the distance between the chosen action and type) to accumulate. In addition, this accumulation is particularly quick for low types because they experience little distortion making their material utility insensitive, and so the action needs to increase by a lot in order to provide sufficient deterrence. Thus, for a small threshold type t, moving from an allocation with initial separation to one with a floor replaces an excessively high action induced by the above "rat race" effect with the more moderate reputational loss from pooling with lower types.

Specializing to the exponential model in Section 4 makes the set of continuing allocations and the minimized continuation loss independent of the initial type t. In addition, the continuation loss admits a recursive structure: the first pooling interval pins down both the loss for the principal on this first interval and the initial loss for the first threshold. Proposition 2 shows that a set of continuation losses that solve the associated Bellman equation must be optimal. I use this result to establish Theorem 2, i.e. that separating is eventually optimal. The intuition reverses that for the optimal floor: (i) for high types where distortion is already large, separating types increases loss very gradually relative to pooling, and (ii) unlike with the floor, the action for any other pooled set has to respect the incentive constraint for some lower type.

1.1. Related Literature

The delegation literature was initiated by Holmstrom (1984) which, assuming interval delegation, finds that it is optimal to cap against the agent's bias. Alonso and Matouschek (2008) and Amador and Bagwell (2013) study generalizations of this standard model and reach interval delegation as a conclusion under palatable assumptions on the preferences and distribution of types. Under some assumptions, Kovac and Mylovanov (2009) shows that these features are robust to allowing for stochastic mechanisms. Dessein (2002) and Melumad and Shibano (1991) also study the original model and compare delegation to alternative communication protocols.

A number of papers study different delegation technologies without changing the material bias of the agent. Krishna and Morgan (2008) and Ambrus and Egorov (2017) study the delegation model with transfers or money burning. Armstrong and Vickers (2010) study an agent who has private information over which choices are available. Frankel (2014) studies a delegation model with many decisions and takes a worst case approach. Halac and Yared (2020) adds the possibility for the principal to verify the agent's type at a cost.

My paper is also related to the large literature dealing with career concerned agents initiated by Holmstrom (1999), which studies a model where the outcome can be observed. My paper fits better into an alternative strand which assumes that only the agent's choice can be observed. Examples include Morris (2001), Prendergast and Stole (1996), and Scharfstein and Stein (1990). Visser and Swank (2007), Moscarini (2007), Kartik and Van Weelden (2019) all integrate career concerns into cheap talk models. A closely motivated set of papers is Ottaviani and Sorensen (2006a) and Ottaviani and Sorensen (2006b) which attempt to compare predictions in cheap talk when the agent is motivated by career concerns vs. material concerns.

Similarly, the incentives in the current reputational communication model are related to those in classic costly signaling models, e.g. in Spence (1973). Frankel and Kartik (2019) study a signaling model whose preferences map closer to those in the current model. Many studies ask various design questions in this framework. Zubrickas (2015) and Dubey and Geanakoplos (2010) ask how to optimally pool test scores into grades (the latter examines when test scores are random from the perspective of the student). Relatedly, Saeedi and Shourideh (2026), Hopenhayn and Saeedi (2023), and Horner and Lambert (2021) study how an intermediary can commit to information disclosure policies to increase the quality of goods produced or effort taken.[4] One ostensibly close exercise is that in Onuchic and Ray (2023), which looks at restricting the choice set of education levels available in a costly signaling problem. However, their focus is on issues of a non-common prior between the student and university. They also impose a linear utility for the student over education choices which is downward biased relative to the university, and that the lowest education level must be in the choice set, so the tradeoffs highlighted in this paper are not present.[5]

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[4] See Lizzeri (1999) and Ostrovsky and Schwarz (2010) which study related questions without the moral hazard component.

[5] For example, the separating allocation in their model is independent of the current distortion, removing the key point that distortion accumulates especially quickly for low types.

Kartik (2009) and the costly lying literature also studies signaling tradeoffs similar to that in the current paper. Roughly, the costly message is mapped to the action and the receiver's response to the message is mapped to the reputation. My main results speak to how a designer should limit the set of messages available to the sender in these models.

In broad strokes, the main intuition behind the optimal floor has echoes in the economic theory literature going back to Maskin and Riley (1984) and Rothschild and Stiglitz (1976). The reason why a low value or low risk buyer is sometimes excluded in contracting problems is because respecting these buyer's incentive constraints negatively impacts profits on high value buyers. More recent examples include Fuchs and Skrzypacz (2015), which shows that an initial trade subsidy increases welfare over laissez-faire policies, and Krishna and Morgan (2008) and Karamychev and Visser (2017) who show that using transfers/money burning to incentivize the correct action for high types is sub-optimal.

The use of recursive optimization methods in communication problems is apparently novel. However, the recursive structure of communication problems with monotonic allocations is used to construct cheap talk equilibria in Crawford and Sobel (1982). Deimen and Szalay (2019) specializes to a family of Laplace distributions to simplify this procedure. This is analogous to how specializing to the exponential distribution over the types reduces the dimensionality of finding the optimal delegation set.

2. Model and Preliminary Results

2.1. Setup

Overview There is a principal, an agent, and an outside observer or market. The agent has private information about his type $t \in T \equiv [0, M]$ where $M \in [0, \infty)$. There is a common prior type distribution with density $f : T \to \mathbb{R}^+$ and associated measure F. I assume the density is bounded above and below, i.e. $\exists \overline{k} \geq \underline{k} > 0$ with $f(t) \in [\underline{k}, \overline{k}]$ $\forall t \in T$. The principal has control over an action $a \in A \equiv \mathbb{R}^+$ and delegates a subset of available choices to the agent. The agent chooses an action from this delegation set. The market then observes the chosen action and updates their belief about the agent's type.

Preferences I represent preferences in terms of losses instead of utilities as it proves more convenient throughout. The principal's loss given action choice a and type t is given by $(a - t)^2$. The agent has two components to his preferences, a material component and a reputational component. The material component is the same as the principal's, i.e. there is no material misalignment. Given a belief $\mu \in \Delta T$ held by the market, an agent of type t

has reputational loss given by $\rho(t - \mathbb{E}[t'|t' \sim \mu])$, where $\rho > 0$ is a positive constant weight. The reputational loss is normalized (i.e. it does not affect equilibrium behavior) to be 0 if μ is degenerate on t. Given market belief μ, action a, and type t, the total loss of the agent is given by

$$(a-t)^2 + \rho(t - \mathbb{E}[t'|t' \sim \mu]).$$[6]

Delegation and Equilibrium Focus The principal delegates a set $\tilde{A} \subset A$. The agent's strategy is a mapping from types to distributions over actions in the delegation set: $\sigma : T \to \tilde{A}$.[7] The market forms beliefs over the agent's type given the chosen action, $\mu : \tilde{A} \to \Delta T$. An equilibrium given $\tilde{A}$ is a pair, σ, μ such that $\mu(a)$ is consistent with Bayes rule $\forall a \in \text{Supp}(\sigma)$, and σ minimizes loss for the agent given reputations consistent with μ.

In general, there can be multiple equilibria for the same delegation set $\tilde{A}$.[8] However, many of these equilibria are sustained by unreasonably punitive off-path beliefs, e.g. beliefs that assume that all off-path actions are taken with certainty by $t = 0$. I instead focus on the equilibria satisfying the classic D1 refinement introduced by Cho and Kreps (1987).[9] In Section 5, I show that under a regularity condition on the type distribution, there is a unique equilibrium satisfying the D1 refinement for any delegation set $\tilde{A}$. Moreover, this unique D1 equilibrium uses all delegated actions, i.e. $\tilde{A} = \sigma(T)$.

This result justifies focus on the direct mechanism approach. The principal specifies an incentive compatible **allocation** $x : T \to A$, and uniquely implements it by delegating all used actions, i.e. $\tilde{A} = x(T)$. This implementation also allows one to ignore considerations of off-path beliefs. Outside of the D1 refinement, my analysis is still relevant if one focuses on the principal's preferred equilibrium given any delegation set. Until Section 5, I omit further mention of the delegation set and focus on the allocation $x : T \to A$.

Allocations Given an allocation $x : T \to A$, the market believes the action is chosen via the allocation. Thus, the market's belief over the type after observing action $a \in x(T)$ is

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[6] In Section 6, I discuss how the main results extend to more general material and reputational preferences. I also discuss how the results would change if the principal and agent were misaligned on their material preferences.

[7] Because of the quadratic loss material preferences, it is never optimal for the agent to mix between actions for a positive measure of types. Thus, the restriction to pure strategies is without loss.

[8] This equilibrium multiplicity is only a potential issue for the principal in "one direction": if there is an alternative equilibrium which uses more actions than the intended equilibrium σ, the principal can just shrink the delegation set to $\tilde{A} = \sigma(T)$. The real issue is alternative equilibria that use fewer actions than intended.

[9] See Section 5 for details.

simply the prior conditioned on $x^{-1}(a)$. This means that each allocation induces reputations given by $r_x(a) \equiv \mathbb{E}[t'|t' \in x^{-1}(a)]\ \forall a \in x(T)$. I refer to the realized reputation as $r^*_x(t) \equiv r_x(x(t))$. Given allocation x, the loss to the agent of type t from choosing action $a \in x(T)$, is denoted $L^A(a, t|x) \equiv (a-t)^2 + \rho(t - r_x(a))$. I refer to the realized loss for type t as $L^A(t|x) \equiv L^A(x(t), t|x)$. The expected loss for the principal is denoted $L^P(x) \equiv \int_0^M (x(t') - t')^2 f(t')dt'$. Note that because the agent's preferences over market beliefs are linear, the expected reputational loss is 0 for any allocation x and $L^P(x) = \mathbb{E}[L^A(t|x)]$.

Incentive Compatibility The principal is restricted to choose incentive compatible (IC) allocations. Given the reputations r_x, the agent of each type t must be incentivized to choose their allocated action $x(t)$. That is, $x(t)$ is an equilibrium of the Bayesian game between the agent and the market given choice set $x(T)$. An allocation $x : T \to A$ is incentive compatible on T if

$$L^A(t|x) = \min_{a \in x(T)} L^A(a, t|x) \quad \forall t \in T. \tag{1}$$

Let the set of allocations satisfying (1) be $IC(T)$.[10] The principal seeks to minimize their loss over all incentive compatible allocations. That is, the principal solves

$$\inf_{x \in IC(T)} L^P(x). \tag{2}$$

Discussion One can interpret the reputational concerns of the agent in a couple different ways. The first interpretation is implied by the setup above: the reptutational concern is a reduced form for the agent being compensated in the future based on the market's belief about his type. For example, as stated in the introduction, the type could be correlated with ability and so the market's belief could capture future hiring opportunities.

The reputational concern could also capture the expected gains in future repeated identical interactions with the same principal who lacks dynamic commitment power and can choose to replace the agent. To be succinct, I have not modeled the principal as valuing higher type agents. However, if the principal's loss were instead given by $(a-t)^2 - t$, this would induce an incentive to replace low type agents, and clearly not change the optimal allocation. Through all possible interpretations, it is very important that only the action and not the loss, nor the type of the agent is observed by the party whose beliefs the agent values.

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[10] It will also prove useful to correspondingly refer to the set of allocations that are incentive compatible on some subset of types $S \subset T$ as $IC(S)$.

The model differs from the material delegation framework only through the agent's preferences. The exogenous material bias in the standard model has been replaced by an endogenous reputational bias. The reputational bias still leads the agent to prefer higher actions than the principal. However, the extent to which the agent and principal are misaligned depends on the allocation. To see why the reputational term leads the agent to be upward biased relative to the principal, consider two actions $a_1 < a_2$ in the range of some incentive compatible allocation x. As will be made precise in Lemma 1 below, higher actions must be allocated to higher types, meaning that $r_x(a_2) > r_x(a_1)$. This implies that if the agent of type t is indifferent between a_1 and a_2, the material loss must actually be lower for a_1, i.e. $(a_1 - t)^2 < (a_2 - t)^2$. This is equivalent to saying that the principal strictly prefers to allocate a_1 rather than a_2 to type t. As is discussed in Section 6, preserving this upward biased condition is essential to considering extending the results to more general preferences.

It may seem strange to set up a comparison of these two models that have fundamentally different agent preferences. However, material bias communication models have a lot in common with the reputational bias model in this paper. Section 6 discusses how under cheap talk, the canonical material bias model has the same set of equilibria as that for the current reputational bias model. In addition any incentive compatible allocation in the current reputational model is an IC allocation in *some* material bias delegation model, and vice versa. The key point, as will be seen in the proceeding analysis, is that the *set* of IC allocations is always different between the current reputational delegation model and *any* material delegation model.

2.2. Preliminaries

I first characterize incentive compatible allocations. For any allocation x let $J_x \subset T$ be its set of discontinuities.

Lemma 1. *An allocation $x \in IC(T)$ if and only if*

1. *x is increasing.*

2. *J_x is a countable nowhere dense set. Let $\underline{t}, \bar{t} \in J_x$ such that $(\underline{t}, \bar{t}) \cap J_x = \emptyset$. $\forall t \in (\underline{t}, \bar{t})$, either*

   (a) $x'(t) = 0$ or (pooling)

   (b) $x'(t) = \frac{\rho}{2(x(t)-t)}$ (separating)

3. *$\forall t \in J_x$,*

$$\lim_{t' \to t^+} L^A(x(t'), t|x) = \lim_{t' \to t^-} L^A(x(t'), t|x).$$

Given the characterization, I equate each $x \in IC(T)$ with its right continuous counterpart.[11] Lemma 1 is very similar to the characterization of incentive compatibility in material delegation: it is identical to lemma 1 in Alonso and Matouschek (2008) aside from point (2b).

First, all IC allocations are monotonic. The reason is that for any allocation x, $L^A(a, t|x)$ is strictly submodular in (a, t). This monotonicity means that in each IC allocation, each action is associated with an interval of types, and so the reputations are the associated expectations on these intervals. That is $r_x(a) = \mathbb{E}[t'|\underline{t} \leq t' \leq \bar{t}]$ for some $\underline{t} \leq \bar{t}$. Because of this feature, it will be useful to notate $R(\underline{t}, \bar{t})\mathbb{E}[t'|\underline{t} \leq t' \leq \bar{t}]$ $\forall \underline{t} \leq \bar{t}$. The second part of the result says that any $x \in IC(T)$ segments the type space into countably many intervals on which the allocation is either constant, or a solution to the differential equation in (2b). I refer to these two types of allocations as pooling and separating respectively. While this guarantees local incentive compatibility on each interval, the last part of the lemma guarantees local incentive compatibility at the endpoints.

As mentioned above, the main difference between Lemma 1 and the analogous characterization in material delegation is the behavior of the separating allocation. A separating allocation on a given interval $(\underline{t}, \bar{t})$ is defined by the property that the action reveals the type, i.e. $x^{-1}(x(t)) = t$ or $r_x^*(t) = t$ $\forall t \in (\underline{t}, \bar{t})$. In this case incentive compatibility means that,

$$t \in \text{argmin}_{t' \in T} (x(t') - t)^2 + \rho(t - t').$$

Taking first order conditions gives the differential equation in (2b).

Figure 1 illustrates an example of an IC allocation with $|J_x| = 3$. The left panel displays the action as a function of the type while the right panel displays the corresponding loss for the agent. A few observations are worth noting now. First, the action jumps whenever the allocation switches from pooling to pooling, separating to pooling, or pooling to separating, because in each of these cases there is also a jump in the corresponding reputations. This is in contrast to material delegation wherein the allocation only jumps between two pooling intervals. Second, the loss of the agent is continuous in the type and almost everywhere differentiable. Third, and most importantly, the separating allocation is different than that for material delegation; in particular, the separating loss to the agent is increasing in the type, while in material delegation the agent gets his ideal action on separating intervals. The separating allocation will be important for the analysis and so it will prove useful to further explore (2b).

There are a continuum of increasing solutions to (2b) pinned down by the initial con-

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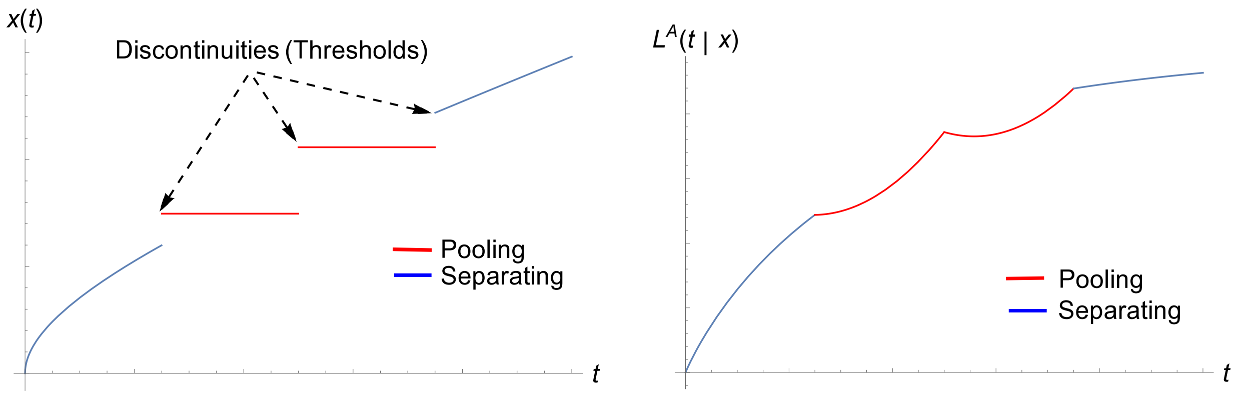
[11] This changes the allocation on a measure zero set of types and thereby does not affect the principal's loss.



(a) (b)

Figure 1: An Incentive Compatible Allocation

dition. Let $d_u(t)$ solve (2b) with initial condition $d_u(0)^2 = u$. Let $D_u(t) \equiv (d_u(t) - t)^2$ be the separating loss given that type $t = 0$ experiences loss u. While $d_u(t)$ does not admit an explicit representation, key properties are derived below. The separating loss for various initial conditions is displayed in Figure 2.

Lemma 2. *Properties of the separating allocation:*

1. *If $u \leq (\geq) \rho^2/4$ then the loss of the agent $D_u(t)$, is increasing and concave (decreasing and convex) in t.*
2. $\forall u \geq 0$, $\lim_{t\to\infty} D_u(t) = \rho^2/4$.
3. *$\forall t$, $D_u(t)$ is strictly increasing in u.*

The first two points report that the separating loss monotonically asymptotes to $\rho^2/4$. At this loss, increasing the action one to one with the type exactly deters the agent from seeking a higher reputation, so loss remains constant. The third point says that the separating allocation is increasing in the initial loss u. This means that the principal's optimal separating allocation is given by $d_0(t)$ and is associated with the lowest curve in Figure 2.[12]

I conclude this section by asserting that a minimum to (2) exists.

Lemma 3. *There exists an allocation $x^* \in IC(T)$ that minimizes $L^P(x)$ across all $x \in IC(T)$.*

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[12] The separating allocation is related to the separating equilibrium in lying cost models. Specifically (2b) is related to the differential equation in Lemma 1 in Kartik (2009).

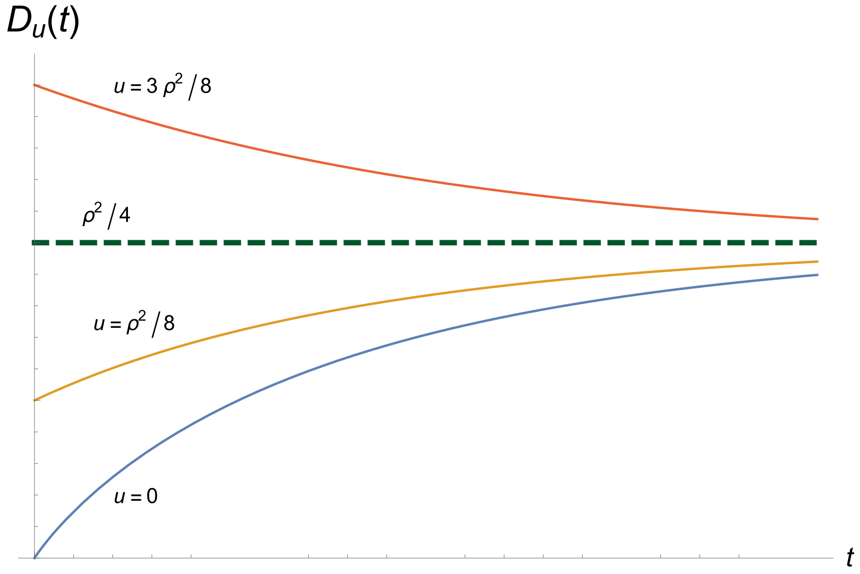


Figure 2: Separating Allocations

3. The Optimality of a Floor

Theorem 1 (Optimality of a Floor). *$\exists K > 0$ such that for every solution x^* to (2), $\exists a_0 \geq K$ with $x^*(t) = a_0\ \forall t \in [0, a_0]$.*

The main result says that it is optimal to restrict the agent's flexibility for low actions in any optimal allocation. Specifically, there is a uniform bound $0 < K \leq x^*(0)$ for any optimum x^*. An implication of this, is that the optimal allocation also pools a non-trivial first interval of types. This is because (as will be clarified below), it is always optimal to set the minimum action lower than the highest type to which it is allocated to, i.e. $x^*(0) < \sup(x^{*-1}(0))$. Another implication is that the floor action is isolated within the delegation set, i.e. $\exists \delta > 0$ such that for every optimal allocation $[0, x^*(0)+\delta] \cap x^*(T) = \{x^*(0)\}$. This is because the next highest action under x^* will have a discontinuously higher reputation than $x^*(0)$, so incentive compatibility necessitates a corresponding jump in the action as well. As made precise in Section 6, these features are unique to delegation with a reputational bias; in delegation to an agent with an upward material bias, it is never optimal to restrict flexibility at the bottom of the type distribution.

It is of course possible to construct IC allocations that violate the condition of Theorem 1. An example is the allocation in Figure 1. As the action for type $t = 0$ is $a = 0$, such alternative allocations perform better than any optimal allocation for low types. Indeed, any optimal allocation sacrifices some loss on low types by setting a floor in exchange for lowering the loss on higher types.

The argument behind this derives from the following thought experiment: given a threshold type $t > 0$ and that the principal is choosing the optimal allocation for types above t, how should the principal select the allocation below t in order to minimize total loss? In general, this consideration involves two terms: the loss below t, and the loss above t. In the absence of a floor though, the threshold t can be taken arbitrarily small, and so the loss below t is unimportant. Thus, in designing the allocation below this threshold t, the important consideration is how this affects the loss above t.

While the proof sketch is in Subsection 3.2, the basic intuition follows two steps: (i) pooling types below t reduces type t's loss relative to any other allocation below t, and (ii) the loss of type t is a proxy for the principal's loss above type t. Point (ii) holds generally and is termed the alignment principle in the next subsection. Some intuition for point (i) can be gleaned by inspecting the separating allocation. The idea is that distinguishing between agent types that experience small losses requires dramatic increases in the action. Consider (2b) which characterizes the separating allocation. In this extreme case in which the principal distinguishes all types, the action increases infinitely fast as distortion – $(x(t)-t)$ – becomes small. Such dramatic increases are necessary because the agent's material loss near his efficient outcome is arbitrarily small relative to the reputational benefit of claiming to be a higher type. This makes "ignoring" the IC constraints of these low types by imposing a floor especially attractive.

3.1. Continuing Allocations and the Alignment Principle

Consider some type $t \in T$, an allocation $x \in IC([0,t])$ below t, and another allocation $y \in IC([t,M])$ above t. Under what conditions can one join these allocations together to form an incentive compatible allocation? That is, when is the allocation defined by

$$z(t') \equiv \begin{cases} x(t') & t < t \\ y(t') & t \geq t \end{cases}$$

an element of $IC(T)$. The answer is given by point 3 of Lemma 1: the agent of type t must be indifferent between x and y. That is, the only information about x needed to specify y is the loss that type t experiences under x. Motivated by this, Let $y : [t,M] \to A$ be a **continuing allocation** at (t,u) if $y \in IC([t,M])$ and $L^A(t|y) = u \geq 0$. Denote the set of these continuing allocations as $C(t,u)$.

Define the **minimized continuation loss** as

$$V(t,u) \equiv \frac{1}{F([t,M])} \inf_{y \in C(t,u)} \int_t^M (y(t')-t')^2 f(t')dt'. \tag{3}$$

Note that if one solves (3) for every (t,u) then this greatly simplifies finding the optimal allocation. To see this, note that the solution to (2) is the solution to

$$\inf_{a_1,t_1} \int_0^{t_1} (a_1-t')^2 f(t')dt' + F([t_1,M])V(t_1,(a_1-t_1)^2+\rho(t_1-R(0,t_1))). \tag{4}$$

That is, the principal only needs to optimize over the first action and first threshold. These choices pin down the agent's loss at the first threshold and thereby associated minimized continuation loss. This is the method used in Section 4. For now I focus on how $V(t,u)$ changes with the initial loss u.

An illustrative example of a continuing allocation is the separating continuing allocation at (t,u) given by $d_u(t'-t)+t \ \forall t' > t$. As derived in Lemma 2 point 3, the principal does better with the separating continuing allocation when the initial loss u is lower. The next proposition shows that this property extends to the minimized continuation loss.

Lemma 4. *$\forall u \geq 0$ and $t \in T$, there exists a solution to* (3).

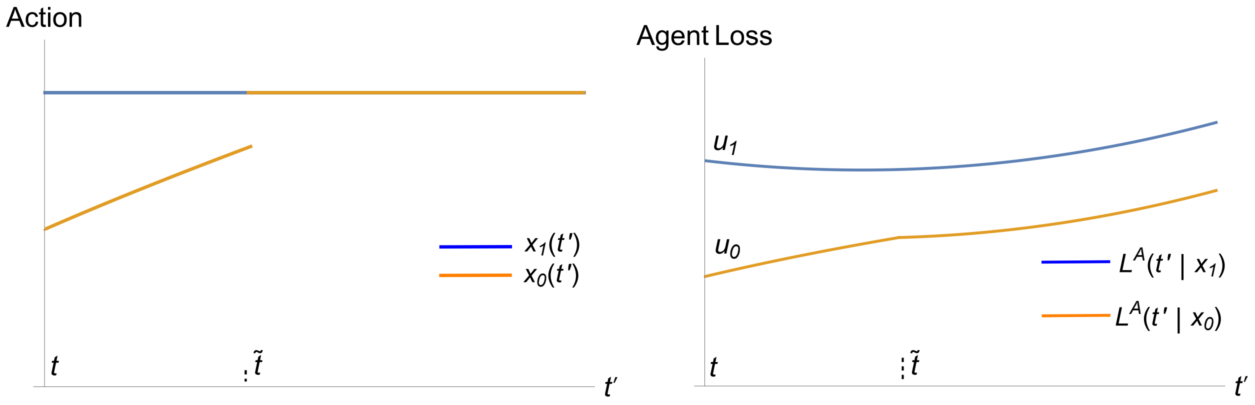
Proposition 1 (The Alignment Principle). *$\exists k > 0$ such that $\forall t \in T$, $\forall u > 0$, $\forall \varepsilon \leq u$,*

$$\frac{V(t,u)-V(t,u-\varepsilon)}{\varepsilon} > k.$$

The alignment principle says that the minimized continuation loss above type t is increasing in the initial loss of type t. Moreover, the magnitude of this change in continuation loss is uniformly bounded away from 0. The implication is that all else being equal, the principal should seek an allocation x below type t that minimizes type t's loss.

The intuition comes from the fact that the reputational concern induces a bias towards higher actions. Reconsider the piecewise allocation z above formed by joining x and y, and suppose that $L^A(t|x) < L^A(t|y)$. Note that by monotonicity, $r_x^*(t) \leq t \leq r_y^*(t)$, i.e. x provides a lower reputation than y to type t. This means that in order to satisfy $L^A(t|x) < L^A(t|y)$, it must be that $(x(t)-t)^2 < (y(t)-t)^2$, i.e. material loss must be less from x than y for type t. This means that the principal would also prefer to assign $x(t)$ rather than $y(t)$ to type t; in other words their preferences are *aligned* at type t.

There are many ways the principal can take advantage of the slack introduced by this alignment. The method used in the proof is illustrated in Figure 3. Consider $u_1 > u_0 \geq 0$, and a continuing allocation x_1 at (t, u_1). The proof constructs a better continuing allocation at (t, u_0) by replacing x_1 on $[t, \tilde{t}]$ with the separating continuing allocation. Here, $\tilde{t}$ is the first type indifferent between this separating allocation and $x_1(\tilde{t})$ given the altered reputations. The left panel of Figure 3 illustrates these two continuing allocations, and the right panel shows the agent's loss for every type between t and the next threshold under x_1. there are two main conclusions from this figure: (i) because the principal's expected loss and the agent's expected loss are equal, this change decreases loss for the principal,[13] and (ii) the next threshold under x_1 has a lower initial loss under the new allocation than under x_1, which means that one can continue this construction across the entire type space. Doing so, yields an allocation with strictly lower loss.[14]



(a)

(b)

Figure 3: The alignment principle

3.2. Proof Sketch of Theorem 1

Consider that Theorem 1 does not hold. This means $\forall \varepsilon > 0$, there is an optimum x^* such that $x^*(0) < \varepsilon$. Let $\tilde{t}$ be the first threshold, i.e $r^*_{x^*}(0) = R(0, \tilde{t})$. The principal's optimal action for this first interval of types $[0, \tilde{t}]$ is $R(0, \tilde{t})$. I first note that that $x^*(0) \in [R(0, \tilde{t}), \tilde{t}]$. Otherwise the principal could improve both her loss on $[0, \tilde{t}]$ by moving $x^*(0)$ into this interval. But such a change would also decrease the loss of the threshold type $\tilde{t}$, and therefore the principal's loss on $[\tilde{t}, M]$ by the alignment principle. Since $R(0, \tilde{t}) \leq x^*(0)$, $\tilde{t}$ can be taken arbitrarily small as well.

It is useful to consider two cases based on whether the "next thresholds" under x^* can

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[13] The improvement is actually type by type.

[14] It is important that the rate of change is strictly positive rather than the weaker property that $V(u) > V(u - \varepsilon)$. Theorem 1 requires a first order change uniformly bounded away from 0.

also be taken arbitrarily small or not.[15] Consider first that the next threshold $\tilde{s}$ is large. In this case, the action for $t \in [\tilde{t}, \tilde{s})$ is respecting the incentive constraint of $\tilde{t}$ at the benefit of decreasing the loss on $t \in [0, \tilde{t})$. However, because $\tilde{t}$ is small, the principal finds it beneficial to get rid of this first interval and freely optimize with respect to the new first action for $t \in [0, \tilde{s})$.

The alternative is that there exist some next threshold $\tilde{s}$ (not necessarily the adjacent one), that is small, i.e. $\tilde{s} < \varepsilon$, but large relative to $\tilde{t}$, i.e. $\tilde{t}/\tilde{s} < \varepsilon$. In this case, one can show that the loss for type $\tilde{s}$ is approximated by that of the minimized separating loss for $\tilde{s}$. Since the loss on $[0, \tilde{s})$ is small and second order for any (reasonable) allocation, this approximation and optimality of x^* implies that the separating allocation must minimize the loss of type $\tilde{s}$ among all allocations. Otherwise, the alignment principle would imply that the principal could improve her loss above $\tilde{s}$ by using some alternative continuing allocation. I use a floor to construct such an alternative allocation: let

$$z(t) \equiv \begin{cases} \tilde{s} & t < \tilde{s} \\ x^*_{\tilde{s},\tilde{u}}(t') & t \geq \tilde{s} \end{cases},$$

where $\tilde{u} \equiv \rho(\tilde{s} - R(0, \tilde{s}))$ and $x^*_{\tilde{s},\tilde{u}}$ is an optimal continuing allocation at $(\tilde{s}, \tilde{u})$. Figure 4 illustrates why z improves on x^*. The left panel illustrates how, under the uniform distribution and for small $\tilde{s}$, type $\tilde{s}$'s loss is lower under x than under the separating allocation. This comparison is general: for small $\tilde{s}$, the separating loss and pooling loss for small $\tilde{s}$ are approximated by $\rho\tilde{s}$ and $\rho\tilde{s}/2$ respectively. The right panel illustrates how z improves loss above $\tilde{s}$ relative to x^* when $x^*_{\tilde{s},\tilde{u}}$ is the separating continuing allocation. While the type by type comparison in loss is specific to the separating continuing allocation, the alignment principle shows that the comparison in expected loss holds generally.

4. The Exponential Model

The previous section showed that a floor is always optimal. The proof sketch makes use of the special properties of the first interval, i.e. that the corresponding action is unconstrained. However, another valid interpretation (at this point) is that pooling, i.e. discretely segmenting all types, is preferred in the reputational delegation framework.

Rebutting this alternative interpretation requires more details about the solution. However, standard methods used to solve the general material delegation model, e.g. those

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[15] Formally, the division in the proof is as follows. Case 1: $\exists \tilde{s} \in J_{x^*}$ and $b > 0$ such that $(\tilde{t}, \tilde{s}) \cap J_{x^*} = \emptyset$ and $\tilde{s} > b$. Case 2: $\exists \tilde{s} \in J_{x^*}$ such that $\tilde{s} < \varepsilon$ and $\tilde{t}/\tilde{s} < \varepsilon$.

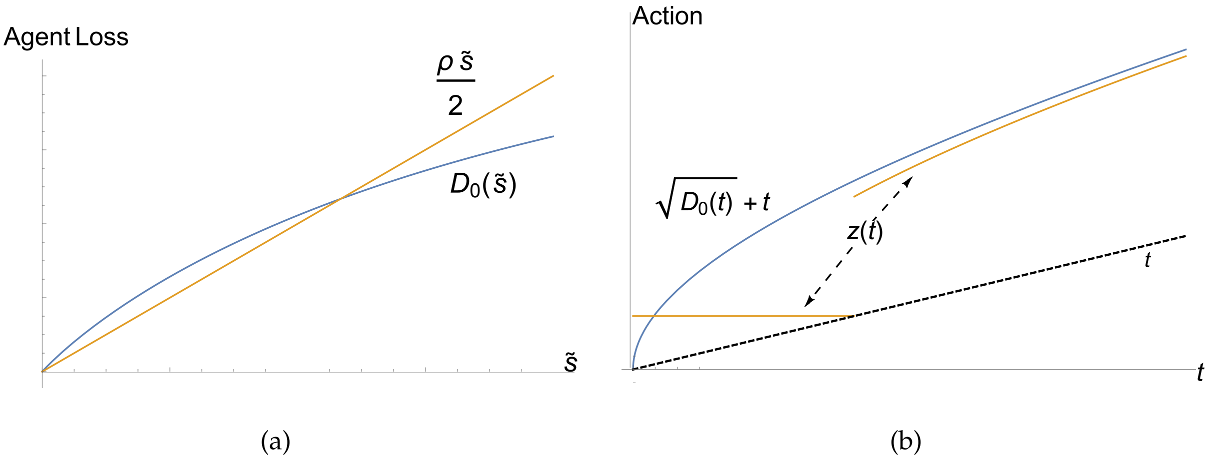

Figure 4: Improving on x^* Using a Floor.

used in Amador and Bagwell (2013), and Kleiner et al. (2021), do not work in the reputational delegation framework.[16] Instead, I specialize to an exponential type distribution in this section, and solve the problem using recursive methods. That is, for the remainder of this section, I assume that $T = [0, \infty)$ and $f(t) = \lambda e^{-\lambda t}\ \forall t$ with $\lambda > 0$.[17] The main takeaway is that the separating allocation is "eventually" optimal, i.e. there exists some type above which the optimal allocation is separating. This means that pooling is not good per-se in reputational delegation, rather it is specifically optimal for low types.

The memorylessness property of the exponential distribution affords key simplifications. The set of continuing allocations and thereby the minimized continuation loss at (t, u) do not depend on the initial type t. Given this fact, I normalize the initial type to 0, and denote a continuing allocation as $y : [0, \infty) \to A$, with $y \in IC(T)$, and $L^A(0|y) = u$. Furthermore, I abuse notation and write $C(t, u) \equiv C(u)$, $V(t, u) \equiv V(u)$, and $R(t_1, t_2) - t_1 \equiv R(t_2 - t_1)$. That is, the minimized continuation loss at u is redefined as,

$$V(u) \equiv \inf_{y \in C(u)} \int_0^\infty (y(t') - t')^2 \lambda e^{-\lambda t'} dt'.$$

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[16] These papers optimize over the entire allocation subject to the envelope version of the incentive constraint. They find Lagrange multipliers under which this problem is convex and use first order conditions to find the optimal allocation. In the current problem, each allocation induces reputations which (i) factor into the envelope constraints and (ii) are expectations over the inverse mapping of the allocation and thereby do not change smoothly with the allocation.

[17] The compactness of the type space is used in the existence results in earlier sections. In this section, I will prove that the optimal allocation exists using a separate argument.

Moreover, the principal's problem is equivalent to

$$\inf_{t_1 \geq 0, a_1 \geq 0} \int_0^{t_1} (a_1 - t')^2 \lambda e^{-\lambda t'} dt' + e^{-\lambda t_1} V\left((a_1 - t_1)^2 + \rho(t_1 - R(t_1)\right). \tag{5}$$

The goal of this section will be to derive properties of the optimal continuing allocations at each u.

4.1. Recursive Formulation

Let y be a continuing allocation at u with first threshold t. This first threshold pins down the reputation for this interval as $R(t)$. Since $u = L^A(0|y) = y(0)^2 - \rho R(t)$, this first threshold also pins down the first action $y(0)$. Finally, $y(0)$ and $R(t)$ together pin down the loss of the first threshold t, i.e., $L^A(t|y)$. Motivated by this, define

$$\begin{aligned} a(t,u) &\equiv \sqrt{u + \rho R(t)} \text{ and} \\ \overline{u}(t,u) &\equiv (a(t,u) - t)^2 + \rho(t - R(t)). \end{aligned}$$

The minimized continuation losses must choose this first threshold optimally.[18]

$$V(u) = \inf_{t \geq 0} \int_0^t \lambda(a(t,u) - t')^2 e^{-\lambda t'} dt' + e^{-\lambda t} V(\overline{u}(t,u)) \quad \forall u. \tag{6}$$

This is a Bellman equation with both a one dimensional state variable – u, and a one dimensional control variable – t. The natural question is whether the converse to the above implication holds: if a set of continuation losses satisfy (6), then are these continuation losses minimized? The next result answers a stronger version of this question affirmatively.

Proposition 2. *Take $\underline{u} \leq \rho^2/4$ and $\{y_u\}$ be a set of continuing allocations at each $u \geq \underline{u}$. Suppose that $\forall u \geq \underline{u}$ $L^P(y_u)$ is differentiable in u and $\frac{dL^P(y_u)}{du} \leq 1$.*

$$\begin{aligned} L^P(y_u) &= \min_{t \geq 0} \int_0^t \lambda(a(t,u) - t')^2 e^{-\lambda t'} dt' + e^{-\lambda t} L^P(y_{\overline{u}(t,u)}) \quad \forall u \geq \underline{u} \qquad (7) \\ \implies L^P(y_u) &= V(u) \quad \forall u \geq \underline{u}. \end{aligned}$$

It is worth noting a few aspects of Proposition 2. First and most salient, one only needs to verify the recursive condition for $u \geq \underline{u}$. This is due to the following regularity property

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[18] The objective takes the form of an infimum instead of a minimum to allow for the allocation which pools all types, i.e. taking the first threshold $t \to \infty$.

of the exponential model.

Lemma 5. *In the exponential model, if* $u \leq \rho^2/4$ *then* $\forall t > 0,\ \overline{u}(t,u) > u$*, and if* $u \geq \rho^2/4$ *then* $\forall t > 0,\ \overline{u}(t,u) > \rho^2/4$.

That is, if initial loss $u \leq \rho^2/4$, then no subsequent threshold can admit a loss less than u. This means that the continuing allocations at losses less than u do not have any impact on the choice of continuing allocations at losses greater than u. This simplification will be especially useful in the next section.

The second caveat is that, unlike in analogous dynamic optimization results, one cannot "ignore" the continuing allocations, and simply check that an arbitrary function $\tilde{V}(u)$ satisfies (7). For example, consider $\tilde{V}(u) = 0\ \forall u \geq 0$. This satisfies (7) by simply choosing $t = 0$, but 0 is clearly not an attainable continuation loss for any u. For this reason, Proposition 2 requires that the continuation losses correspond to actual continuing allocations.[19]

Finally, the continuation losses are required to have a derivative less than 1. This condition guarantees an approximation result in the proof. One may be concerned that this condition makes the result vacuous, but any set of optimal continuing allocations must also satisfy this condition. To see this, let y_u be a continuing allocation at u. Now consider constructing another continuing allocation at $u + \varepsilon$ by maintaining the same set of thresholds as y_u. This means that the change in loss is only the effect of changing the action on each interval to match an increase in the constraint in the initial loss. One can show that the change in loss degrades in the type, i.e. for two types $t'' > t'$, $\frac{d\, L^A(t'|y_u)}{d\, u} > \frac{d\, L^A(t''|y_u)}{d\, u}$. Since $\frac{d\, L^A(0|y_u)}{d\, u} = 1$, by the definition of a continuing allocation, the expectation over these changes is less than 1. Clearly an optimal set of continuing allocations would change the thresholds with u to further decrease the principal's loss,[20] and so this derivative assumption does not restrict the set of allocations.

Proposition 2 provides for the "guess and check" method to solve for the optimal continuation loss. One can conjecture a set of continuing allocations and check that its associated continuation losses satisfy (6). This is the approach taken in the next subsection.

4.2. Separating in the Exponential Model

A focal continuing allocation at u is the separating continuing allocation d_u introduced in the previous section.

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[19] In general choosing $t = 0$, will always guarantee that the RHS of (6) is weakly greater than the LHS. This means that loss function iteration can only work to revise the losses downward.

[20] The argument is actually exact under an optimal set of continuing allocations because any change in the thresholds has a 0 effect on $V(u)$ by an envelope theorem argument.

Theorem 2. *In the exponential model,* $u \geq \rho^2/16 \implies V(u) = L^P(d_u)$.

The proof of Theorem 2 shows that for $u \geq \rho^2/16$, setting $y_u = d_u$ satisfies (7). While the result says that there exists initial losses such that separating is optimal, it does not speak to whether separating will actually arise in the optimal allocation. However, using logic similar to that behind Lemma 5, one can deduce that loss increases throughout the type space until $u \geq \rho^2/4$. This means that, as long as there is an unbounded sequence of thresholds in the optimal allocation (i.e. there is no eventual pooling), then the optimal allocation will eventually be separating. This can be shown analytically for small relative biases.[21]

Lemma 6. *If* $\lambda\rho \leq 4$*, the pooling continuing allocation at* u *given by* $x(t) = \sqrt{u + \rho/\lambda} \equiv p_u \; \forall t \in T$ *is not optimal* $\forall u$.

Corollary 1. *An optimal allocation exists. If* $\lambda\rho \leq 4$*, then there exists* $\bar{t}$ *such that for any optimal allocation* x^*, $\exists \bar{u}$ *with* $x^*(t) = d_{\bar{u}}(t - \bar{t}) + \bar{t} \;\; \forall t \geq \bar{t}$.

Corollary 1 first restates the existence of an optimal allocation. Because the type space is now unbounded, one cannot immediately apply the methods to prove existence from the previous sections. I first establish the fact that the allocation is separating or completely pooling after some $\bar{t}$. Then standard methods imply an optimal allocation on the compact space $[0, \bar{t}]$ given the separating or pooling continuation loss above $\bar{t}$.

While the argument for Theorem 2 is complicated, a broad intuition is as follows. There are two main reasons why a floor is optimal and in particular does better than the lowest separating allocation d_0. First, the first pooling action does not need to respect any incentive constraint to the left and can be set freely, whereas $d_0(t)$ is constructed to equalize the incentive to obtain a higher reputation with the cost of increasing material loss. Second, the separating action $d_u(t)$ increases infinitely quickly for low types.

Neither of these reasons are present when determining the continuing allocation for large initial losses. First, any continuing allocation at u must respect the left incentive constraint that the initial loss is u. Second, the loss under the separating continuing allocation increases very slowly for large initial losses. As shown in Figure 2, at initial losses close to $\rho^2/4$ the separating loss is near constant in the type. This means that the comparison between any continuing allocation which pools some first set of types and the separating allocation is more favorable to the latter at large initial losses.

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[21] This is an analytical limitation rather than reflecting that completely pooling is optimal for some parameters. The numerical methods in Subsection 4.4 never reveal completely pooling as optimal.

4.3. The Uniform Limit

The separating continuing allocation is not optimal for small initial losses. More generally, it is difficult to analytically solve for the optimal continuing allocation at every u. This is partly because there is no explicit representation of the separating continuation loss, which by Theorem 2 will be optimal for large u. However, finding $V(u)\ \forall u$ is not necessary to solve for the optimal allocation. If one can instead show that the floor (t_0, a_0) in (4), must be optimally set so that the loss at the first threshold – $(a_0 - t_0)^2 + \rho(t_0 - R(t_0))$– is greater than $\rho^2/16$, then the optimal continuing allocation is separating by Theorem 2. This requires bounding $V'(u)$ for small initial losses. This is tractable in the uniform limit, i.e. when $\lambda \to 0$. The result, reported below, is that the optimal allocation uses a floor and then separates thereafter. The optimal allocation in the uniform limit is illustrated in Figure 5.

Proposition 3. *There exists $c > 0$ such that for $\lambda \leq c$, the optimal allocation is given by*

$$x^*(t) \equiv \begin{cases} a_0 & t < t_0 \\ d_{\tilde{u}}(t - t_0) + t_0 & t \geq t_0 \end{cases},$$

where $\tilde{u} \equiv (a_0 - t_0)^2 + \rho(t_0 - R(t_0)) > \rho^2/16$, and a_0, t_0 solve,

$$\min_{a_0, t_0} \int_0^{t_0} (a_0 - t')^2 \lambda e^{-\lambda t'} dt' + L^P(d_{\tilde{u}}) e^{-\lambda t_0}.$$

In addition,

$$\lim_{\lambda \to 0} a_0 \to k_1 \rho$$
$$\lim_{\lambda \to 0} t_0 \to k_2 \rho$$

where $k_1 < k_2$ are constants.[22]

4.4. Numerical Solution

Getting an explicit solution for the exponential model is difficult because the separating continuing allocation and thereby the separating continuation loss do not have explicit representations. The previous section shows that the solution admits a single pooling interval and then fully separates thereafter when λ is small. While the optimization in (6) can be analytically difficult, it turns out to be numerically simple. This section numerically

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[22] Specifically, $k_1 \equiv \frac{\sqrt{17}-1}{16}$, $k_2 \equiv \frac{5\sqrt{17}-13}{32}$, and $k_3 \equiv (k_2 - k_1)^2 + k_2/2$.

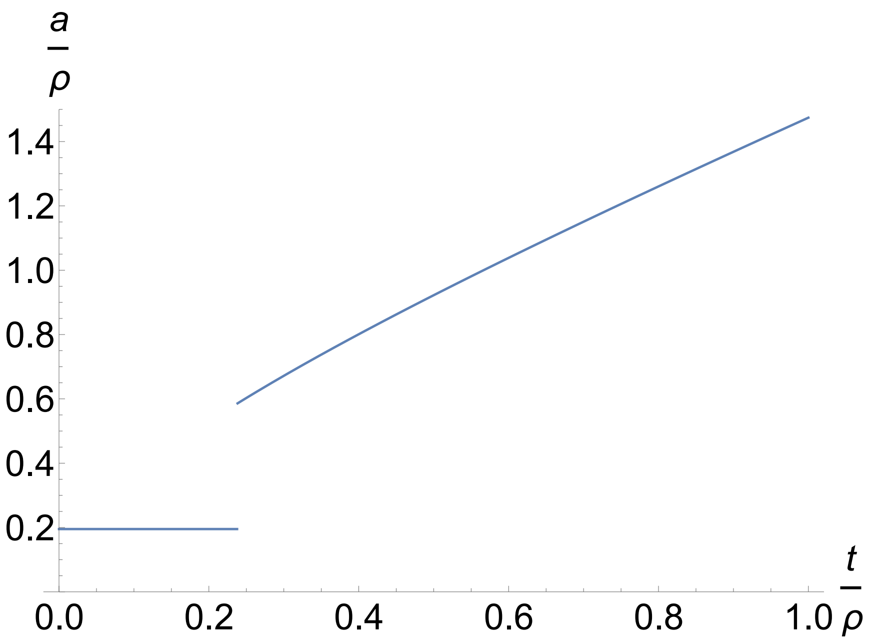

Figure 5: Optimal Allocation in the Uniform Limit.

solves for the optimal allocation for certain parameterizations and shows that these optima mirror the behavior of the uniform limit.

First I make the observation that the problem can be reduced to one with a single parameter. One can show that the minimized continuation loss $V(u)$ in a problem with parameters ρ and λ is the same as that for one with parameters $\rho' = 1$ and $\lambda' = \lambda\rho$ multiplied by the constant ρ^2.[23] This simplification is also represented in the uniform limit solution as the optimal first action and threshold are linear in ρ. With this simplification, one can solve for the optimal allocation for the class of problems in which $\lambda\rho$ is a constant.

The key simplicity is that solving for the minimized continuation loss recursively happens in only one step. The first step starts from the separating continuing allocation d_u as a conjecture, i.e. $\forall u \geq 0$, $y_{0,u}(t) = d_u(t)$ and $V_0(u) \equiv L^P(d_u)$. For each $u \geq 0$, one can solve

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[23] To see this, normalize $\tilde{t} \equiv t/\rho$, $\tilde{u} \equiv u/\rho^2$, and $\tilde{\lambda} \equiv \lambda\rho$. Now define $\tilde{R}(\tilde{t}) \equiv \mathbb{E}[t'|t' \leq \tilde{t},\ t' \sim \tilde{\lambda} e^{-\tilde{\lambda} t'}]$, $\tilde{a}(\tilde{t},\tilde{u}) = \sqrt{u + \tilde{R}(\tilde{t})}$, and $\tilde{u}(\tilde{t},\tilde{u}) \equiv (\tilde{a}(\tilde{t},\tilde{u}) - \tilde{t})^2 + (t - \tilde{R}(\tilde{t}))$.

$$V(u) = \inf_{t \geq 0} \int_0^t (a(t,u) - t')^2 \lambda e^{-\lambda t'} dt' + e^{-\lambda t} V(\overline{u}(t,u))$$

$$\iff V(u) = \rho^2 \inf_{\tilde{t} \geq 0} \left(\int_0^t (\tilde{a}(\tilde{t},\tilde{u}) - t')^2 \tilde{\lambda} e^{-\tilde{\lambda} t'} dt' + e^{-\tilde{\lambda} t} V(\tilde{u}(\tilde{t},\tilde{u})) \right).$$

for the ith iteration by finding

$$t_i^*(u) = \text{argmin}_{t\geq 0} \int_0^t \lambda(a(t,u)-t')^2 e^{-\lambda t'}dt' + e^{-\lambda t}V_i(\overline{u}(t,u)) \quad \forall u,$$

and then setting the continuing allocation

$$y_{i,u}(t) \equiv \begin{cases} a(t_i^*(u),u) & t < t_i^*(u) \\ y_{i-1,\overline{u}(t_i^*(u),u)}(t - t_i^*(u)) & t \geq t_i^*(u) \end{cases},$$

and the continuation loss $V_i(u) = L^P(y_{i,u})$.[24]

Numerically, it turns out that $V_i(\cdot) = V_{i+1}(\cdot)\ \forall i \geq 1$. By Proposition 2, this means that $y_{1,u}$ is an optimal continuing allocation for every u. Figure 6 displays this recursive structure for every $i \geq 1$. First note that by Theorem 2, $t_i^*(u) = 0\ \forall u \geq 1/16,\ \forall i \geq 0$, i.e. the separating continuing allocation is optimal. Indeed, this is the limit of the region where the separating continuing allocation is optimal as $\lambda\rho \to \infty$. This region tends to become larger as $\lambda\rho$ decreases.[25]

The important feature of Figure 6 is that the optimal first threshold always induces a loss at which separating is optimal. That is, the continuation loss relevant for the first threshold choice is always the first separating conjecture $V_0(u) = L^P(d_u)$.

Remark 1. The convergence of the minimized continuation loss after one step is driven by intuitive (and numerically "verifiable") properties of the problem that are again difficult to demonstrate analytically. Roughly the idea is to formulate the problem in (6) as choosing a next threshold state variable $\tilde{u} \geq u$ instead of choosing the first threshold itself, i.e.

$$V_i(u) = \inf_{\tilde{u}\geq u}\ \tilde{V}_i(u,\tilde{u}) \equiv \int_0^{\tilde{t}} (a(\tilde{t},u)-t')^2\lambda e^{-\lambda t'}dt' + e^{-\lambda\tilde{t}}V_i(\tilde{u}),$$

where $\tilde{t}$ is uniquely defined as $\overline{u}(\tilde{t},u) = \tilde{u}$ for $u \leq 1/16$. It seems that $\tilde{V}_i(u,\tilde{u})$ has the following single crossing property for $i = 0$ and $i = 1$: $\forall \tilde{u}_1 < \tilde{u}_2,\ \forall u' \geq u$,

$$\tilde{V}(u,\tilde{u}_2) - \tilde{V}(u,\tilde{u}_1) \geq 0 \implies \tilde{V}(u',\tilde{u}_2) - \tilde{V}(u',\tilde{u}_1) > 0.$$

This property gives that the optimal next threshold loss $\tilde{u}^*(u)$ is decreasing in u if the con-

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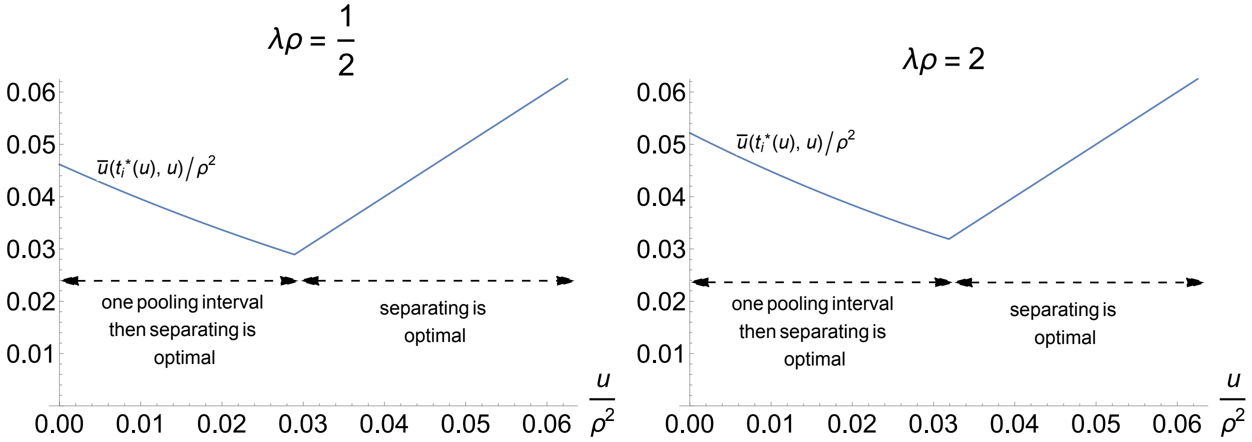
[24] If there are multiple solutions, take $t_i^*(u)$ to be the maximum solution. This is important because if $V_i(u) = V_{i+1}(u)$, then $t = 0$ is always a solution at the $i+1$th iteration. If the Bellman is minimized as $t \to \infty$ set $t_i^*(u) = \infty$.

[25] As proved in the appendix, as $\lambda\rho \to 0$, the region where separation is optimal converges to $u \geq \rho^2/36$.

straint that $\tilde{u} \geq u$ does not bind. Now consider the lowest initial loss u such that separating is optimal $\forall u' \geq u$. Such a u is guaranteed by Theorem 2. Because the RHS of (6) is continuously differentiable, it must be that the constraint that $\tilde{u} \geq u$ does not bind at u. This means that with the aforementioned single crossing property, $\tilde{u}^*(u') \geq u \; \forall u' \leq u$. That is, the separating continuing allocation is optimal at every $\tilde{u}^*(u)$, and $V_i = V_1 \; \forall i \geq 1$.



(a) (b)

Figure 6: The agent's loss at the optimal first threshold.

As mentioned earlier, solving for the optimal continuing allocations at each u is one step away from solving for the optimal allocation. In particular given $V_1(u) = V(u)$, one needs to solve for the first threshold and action a_0, t_0 in (5). In theory, this could mean that even though each optimal continuing allocation has at most one pooling interval, the optimal allocation could have two pooling intervals. However, this does not turn out to be the case. Figure 7 shows the optimal allocations for two parameter specifications below. The optimal allocation has one pooling interval in each case, that is, a floor and then separating is optimal just like in the uniform limit.

The fact that there is one pooling interval in the optimal allocation means that for the optimal choice of a_0, t_0 in (5) means that the separating continuing allocation is optimal at the first threshold loss, i.e. $V(u_0) = L^P(d_{u_0})$ where $u_0 \equiv (a_0 - t_0)^2 + \rho(t_0 - R(t_0))$. Indeed, one can show that at the optimum $u_0 > \rho^2/16 \; \forall \lambda, \rho > 0$, so this conclusion is delivered by Theorem 2. One reason for this is that the incentive to increase the first threshold loss is greater in (5) as compared to that in the continuing allocation problem for any u. This in turn is due to the fact that in (5) the principal sets a_0 optimally as opposed to it being determined exogenously in (6) as $a(t, u)$. The increase in $a(t, u)$ when the first threshold t increases incurs an extra source of loss for the principal in the continuing allocation problem relative

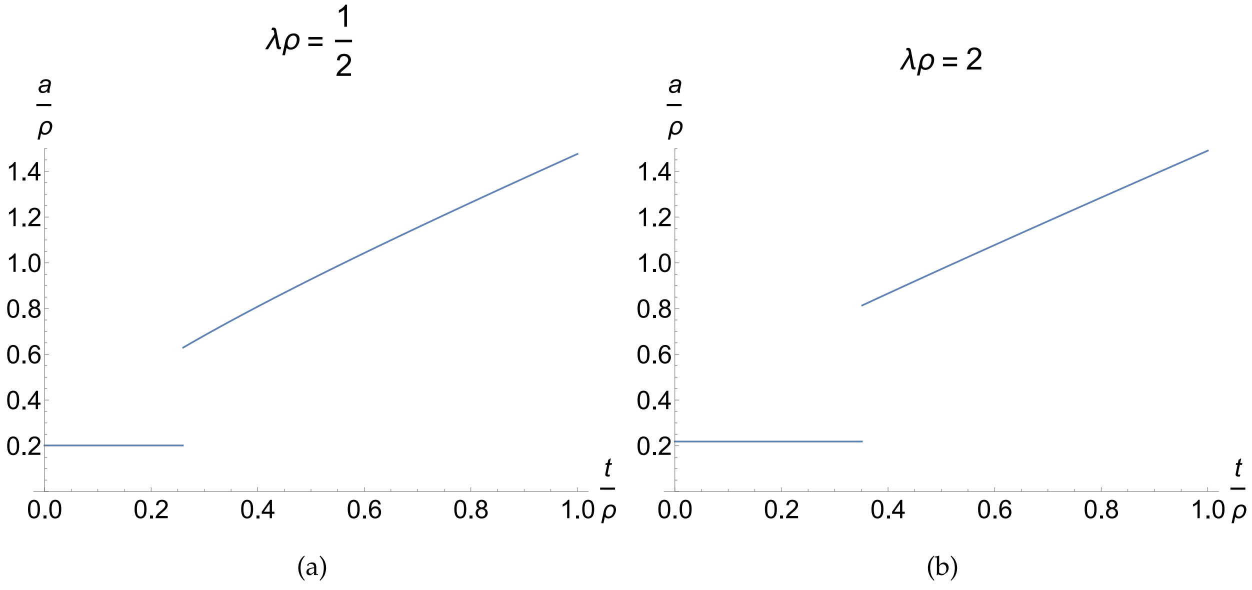


Figure 7: The Optimal Allocation.

to the determination of the first threshold in (5).

5. Unique Implementation

For expositional purposes, the paper focuses on the allocation $x : T \to A$. This ignores questions of equilibrium multiplicity, and how strongly the principal can implement his preferred allocation through delegation. In this subsection I show that, under a regularity condition on the distribution of types, the principal can implement (almost) any IC allocation as the unique equilibrium that satisfies the D1 criterion introduced by Cho and Kreps (1987).

Given a delegation set $\tilde{A}$ and an intended IC allocation $x : T \to \tilde{A}$. There are two sources of equilibrium multiplicity that may cause concern: (i) there may exist alternative incentive compatible allocations that induce the same set of chosen actions but allocate them to different types, and/or (ii) there may exist incentive compatible allocations which induce a strict subset of $\tilde{A}$. In order to address the second issue, one must explicitly consider off path beliefs. This consideration was irrelevant in the main text analysis as only equilibria which used all delegated actions were considered.

An allocation is now a pair $(x, \tilde{A})$ where $\tilde{A} \subset A$, and $x : T \to \tilde{A}$. An incentive compatible allocation is $(x, \tilde{A})$ such that there exists $\mu : \tilde{A} \setminus x(T) \to \Delta T$ such that $\forall t \in T$,

$$\begin{aligned} &L^A(t|x) = \min_{t' \in T} L^A(x(t'), t|x), \text{ and} \\ &L^A(t|x) \geq (a-t)^2 + \rho(t - \mathbb{E}[t'|t' \sim \mu_a]) \quad \forall a \in \tilde{A} \setminus x(T). \end{aligned}$$

This definition puts no structure on the off-path beliefs, and as a result many incentive compatible allocations $(x, \tilde{A})$ with $x(T) \subsetneq \tilde{A}$ may exist. However, these equilibria make use of punitive off-path beliefs, e.g. assuming that any unchosen action is taken by type $t = 0$. I focus instead on beliefs that satisfy the D1 refinement which has proven useful and reasonable in signaling games like that induced between the agent and the market. Consider an allocation $(x, \tilde{A})$. In this context, the D1 refinement says that given $a \in \tilde{A} \setminus x(T)$, if $\exists t', t''$ such that

$$\begin{aligned} &\{\tilde{R} \in [0, M] : \ L^A(t'|x) \leq (a-t')^2 + \rho(t' - \tilde{R})\} \\ \subsetneq &\{\tilde{R} \in [0, M] : \ L^A(t''|x) < (a-t'')^2 + \rho(t'' - \tilde{R})\}, \end{aligned} \tag{8}$$

then the off path belief has $t' \notin \text{Supp}(\mu(a))$. That is, if deviation to an off-path action is tempting for a strictly larger set of off-path beliefs for one type than another, then the market should weight the former type arbitrarily highly relative to the latter type.

In order to deal with equilibrium multiplicity over allocations that use the same set of actions, I show that a regularity condition on the distribution of types rules out this kind of non-uniqueness.

Definition 1. The distribution satisfies condition (M^*) if $\forall a_1 < a_2$, and $\forall t_1 < t_2$ the expression

$$\big((a_2 - t)^2 - \rho R(t, t_2)\big) - \big((a_1 - t)^2 - \rho R(t_1, t)\big) \tag{9}$$

is single crossing from above in $t \in [t_1, t_2]$.[26,27]

To interpret condition (M^*) consider that an allocation splits an interval of types $[t_1, t_2]$ by assigning $[t_1, t)$ to action a_1 and $[t, t_2]$ to action a_2. Condition (M^*) says that moving the threshold type t rightward can never make that type switch from preferring the high action to preferring the low action. From a material loss standpoint, moving the threshold type t rightward results in a stronger preference for the higher action a_2. Thus condition

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[26] A function $g : T \to \mathbb{R}$ is single crossing from above if $\forall t_1 < t_2$ $g(t_1) \leq 0 \implies g(t_2) < 0$.

[27] If $T = [0, \infty)$ is infinite, I also impose the condition to hold for "$t_2 = \infty$".

(M^*) imposes that the changes in reputational differences never overwhelm this material change. A sufficient condition for condition (M^*) is that the density is weakly decreasing so that $R(t, t_2) - R(t_1, t)$ is increasing in t. Thus two examples that satisfy the condition are the uniform distribution and the exponential distribution. However, the condition also permits distributions such that the $R(t, t_2) - R(t_1, t)$ does not decrease too fast relative to the change in material loss.

Lastly, say that $\tilde{A} \subset A$ is a **finite delegation set** if one can write $\tilde{A} = A_1 \cup A_2 \cup ... \cup A_n$, where each $A_i \subset A$ is an (potentially degenerate) interval. Note that restricting to finite delegation sets does not mean delegating a finite set of actions – finite delegation sets can have an arbitrary number of separating regions, i.e. when A_i is a non-degenerate interval.

Proposition 4. *Let the distribution of types satisfy condition (M^*). Let $\tilde{A} \subset A$ be a finite delegation set and $x : T \to \tilde{A}$ be an incentive compatible allocation such that $x(T) = \tilde{A}$. If $(y, \tilde{A})$ is incentive compatible and satisfies the D1 refinement then $y = x$.*

The result says that delegating the range $x(T)$ of an incentive compatible allocation implements this allocation as the unique D1 equilibrium between the agent and the market. The D1 refinement puts a lot of structure on off-path beliefs in this model. In particular, if an action a is off-path under $(y, \tilde{A})$, then the reputation for choosing action a puts probability 1 on $t = \min\{y^{-1}(a') : \ a' > a\}$, i.e. the lowest type who takes an action higher than a. This combined with condition (M^*) gives uniqueness.[28]

To illustrate the result, consider the optimal allocation in the exponential model in the uniform limit. This is spelled out by Proposition 3 as

$$x^*(t) = \begin{cases} a_0 & t < t_0 \\ d_{(\underline{a}-t_0)^2}(t - t_0) & t \geq t_0 \end{cases},$$

where $(\underline{a} - t_0)^2 = (a_0 - t_0)^2 + \rho(t_0 - R(t_0))$. This means that the principal can implement this allocation with the finite delegation set $\tilde{A} = \{a_0\} \cup [\underline{a}, \infty)$. One may be concerned about whether there are other incentive compatible allocations $y : T \to \tilde{A}$ such that $y \neq x$. Proposition 4 and the fact that the exponential distribution satisfies condition (M^*) says that there is no such allocation y that satisfies the D1 refinement. As mentioned there are

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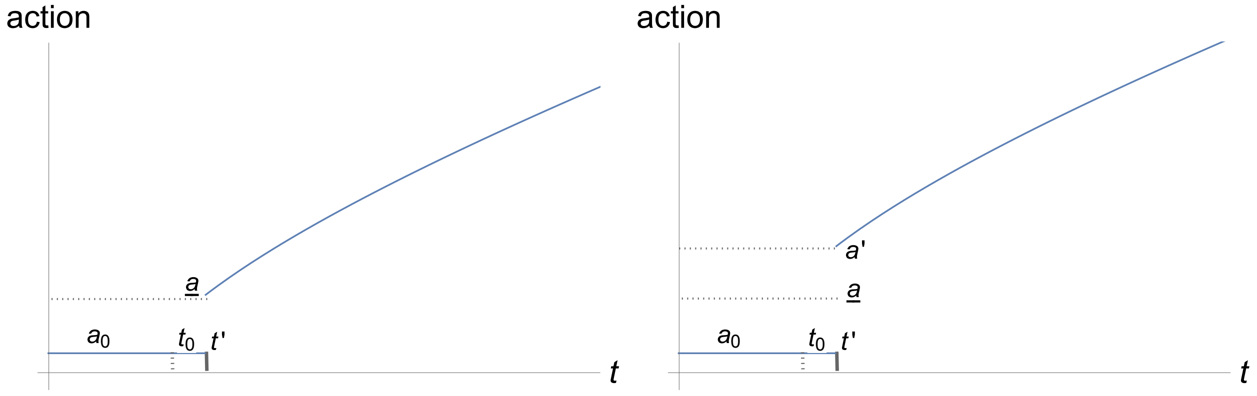
[28] The role condition (M^*) plays in guaranteeing uniqueness is related to that for condition (M) from Crawford and Sobel (1982) (this motivates the name). When $\tilde{A} = \{a_1, a_2, ...a_n\}$ has a finite set of actions, an incentive compatible allocation can be constructed via a difference equation that pins down the "next" threshold as a function of the last two. As pointed out in Crawford and Sobel (1982), this is also true in the standard cheap talk model, and condition (M) there guarantees that there is a unique starting threshold such that the final threshold is the supremum of the type space. The proof of Proposition 4 uses condition (M^*) similarly.

two kinds of non-uniqueness to rule out. Figure 8 below illustrates two associated alternative allocations and why they are not D1 incentive compatible. The left panel displays an alternative allocation that uses all actions, i.e. $y(T) = \{a_0\} \cup [\underline{a}, \infty)$. Whether or not this is incentive compatible comes down to whether there exists another solution besides t_0 to the equation

$$(\underline{a} - t')^2 - (a_0 - t')^2 = \rho(t' - R(t')).$$

However, notice that the right hand side is increasing in t' for the exponential distribution,[29] and the left hand side is decreasing in t', because higher types have relatively lower loss from higher actions. Thus there is at most one solution to this equation. This implies that if we consider an alternative first threshold $t' > t_0$, then t' will strictly prefer $\underline{a}$ over a_0. One potential solution to restore incentive compatibility would be to assign a higher action $a' > \underline{a}$ to type $t' > t_0$, and start the separating continuing allocation from there. The right panel displays such an allocation which uses a strict subset of the actions. However, in this case $\underline{a}$ is off-path and under the D1 refinement commands a reputation of t'. Thus, $\underline{a}$ remains a profitable deviation for t'.



(a) t' prefers $\underline{a}$ to a_0.

(b) t' prefers $\underline{a}$ to a'.

Figure 8: Alternative allocations with the same delegation set.

Remark 2. A weaker version of condition (M^*) is to impose that the expression in (9) is single crossing only for actions in the given delegation set $\tilde{A}$. The proof of Proposition 4 then directly implies uniqueness among all D1 IC allocations $x : T \to \tilde{A}$, and that the unique such allocation uses all actions, i.e. $x(T) = \tilde{A}$. This is especially useful when T is infinite but one knows that the optimal allocation is eventually pooling or separating after some fixed type $\bar{t}$. Let $A_n = [\underline{a}, \infty)$, and $y : T \to \tilde{A}$ be a D1 IC allocation, then $A_n \subset y(T)$.[30]

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[29] This property holds for any log-concave distribution.

[30] This follows from Claim 21, the argument for which says that $y(T) \cap A_n$ is either a singleton, or it includes

This will imply that one only needs to impose condition (M^*) for a compact set of actions in order to get unique implementation of the optimal allocation.

6. Discussion

6.1. Material Delegation vs. Reputational Delegation

A floor being optimal when delegating to an agent with a reputational bias is in stark contrast to optimal delegation to an agent with a material bias. For example consider the canonical material delegation model in which the principal's and agent's loss functions given allocation x are given by,

$$L^A(a,t|x) = (a-(t+b))^2$$
$$L^P(x) = \int_0^M (a-t)^2 f(t)dt$$

with $b > 0$. This is a material delegation model because the agent's loss does not depend on the allocation beyond his type and choice of action. These specific preferences are not arbitrary: the cheap talk game between the principal and the agent, i.e. one in which the principal lacks commitment, has the same equilibrium set as that for the preferences in the main text reputational model if $\rho = 2b$.[31] Alonso and Matouschek (2008) derive that the optimal delegation set for these preferences and a set of "regular" distributions is to fully delegate up to some cap C, i.e. to delegate the set of actions $[0, C]$.

Even if the preferences and distribution do not entail interval delegation to an agent with a material bias, it is never optimal to restrict flexibility for low types as long as the agent is upward biased. To make this point more precise, consider any continuous loss functions

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$[\underline{a}, \tilde{a})$ for some $\tilde{a} > \underline{a}$, and then possibly includes $\max\{A_n\}$. In the case where T is infinite and $A_n = [\underline{a}, \infty)$, there is no maximum action and the rest of the cases are ruled out by the following argument. If $a = \max\ A_n \cap y(T)$, $a + \varepsilon$ has lower material loss than a for high enough types and therefore has infinite reputation under the D1 refinement.

[31] A cheap talk equilibrium in both environments is characterized by its set of thresholds. It is necessary and sufficient for a set of thresholds to constitute a cheap talk equilibrium if each threshold is indifferent between the actions taken for types to the left and right. The action for any message sent by an interval of types $[t_1, t_2]$ is pinned down by receiver optimality as the expectation, i.e. $R(t_1, t_2)$. Let $t_1 < t_2 < t_3$ be three sequential thresholds:

$$\begin{aligned}(R(t_1,t_2)-t_2)^2 + \rho(t_2 - R(t_1,t_2)) &= (R(t_2,t_3)-t_2)^2 + \rho(t_2 - R(t_2,t_3)) \\ \iff (R(t_1,t_2)-(t_2+\rho/2))^2 &= (R(t_2,t_3)-(t_2+\rho/2))^2.\end{aligned}$$

That is, indifference at the thresholds is the same condition for the agent with a reputational bias ρ as for an agent with a material bias $\rho/2$.

for the agent and principal, $l^A : A \times T \to \mathbb{R}$ and $l^P : A \times T \to \mathbb{R}$ that are both strictly submodular in (a, t), strictly convex in a, $\forall t$, and such that $\forall t \in T$, and $\forall a_1 < a_2 \in A$,

$$l^A(a_1, t) \leq l^A(a_2, t) \implies l^P(a_1, t) < l^P(a_2, t).$$

The displayed condition says that the principal always strictly prefers a lower action if the agent weakly prefers it, i.e. the agent is upward biased relative to the principal. Now let $a^*(t) \equiv \arg\min_{a \in A} l^A(a, t)$. For any density f, every optimal allocation has $x(0) \leq a^*(0)$.[32] In addition, it is without loss of optimality to delegate $[0, x(0)) \cup x(T)$, i.e. full flexibility at the bottom is optimal. This means that we cannot find a material delegation model with a purely upward material bias that has the same optimal allocation as the reputational delegation model.

6.2. Alternative Material Preferences

The main analysis is carried out with the assumption of quadratic loss as the form of material preferences of both the agent and the principal. It can be shown that, for Theorem 1, this analysis extends to any convex continuously differentiable loss function of the distance between the chosen action and the type, i.e. $l(|a - t|)$. The key points used in the analysis are that, (i) $l'(0) = 0$, and that (ii) l is shared as the material loss for both the principal and the agent.

Point (i) guarantees that in order to incentivize the agent to separate for low types, the principal must increase the action dramatically in order to deter the agent from seeking a small reputational gain from misreporting. To see this, consider a separating allocation $x : T \to A$ under such a general loss function. According to the analysis in Section 2, it is analogously pinned down by an initial loss condition and the following differential equation,

$$x'(t) = \frac{\rho}{l'(|x(t) - t|)}.$$

Because $l'(\varepsilon) \approx 0$ for small ε, when the agent is experiencing small losses, deterring a small reputational gain requires an arbitrarily large increase in the action.

Point (ii) is sufficient in guaranteeing the alignment principle– Proposition 1– which is

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[32] To see why, suppose that an optimal allocation has $x(0) > a^*(0)$. Now take the type $\tilde{t}$ to be the maximum type such that $a^*(\tilde{t}) \leq x(0)$. Note that by incentive compatibility $x(t) = x(0)\ \forall t \in [0, \tilde{t}]$. Since the agent has an upward bias, the following incentive compatible allocation improves on x:

$$y(t) \equiv \begin{cases} a^*(t) & t < \tilde{t} \\ x(t) & t \geq \tilde{t} \end{cases}.$$

key to the intuition of an optimal floor. The alignment principle relies on the fact that the agent is upward biased relative to the principal, i.e. if an agent type is indifferent between two actions in an allocation then the principal will prefer that the lower action be assigned to that type. Given that the agent and principal share the same material preference, the agent preferring to be seen as a higher type, i.e. his reputational bias, is sufficient to induce this kind of upward bias. If the agent's material preferences were negatively biased relative to the principal the agent may no longer have a net upward bias, and the alignment principle may not go through.

The main text analyzes the case of no material bias to obtain the starkest comparison between existing results. However, the above paragraph suggests that the current analysis can deal with misaligned material preferences between the principal and agent if it in fact reinforces the agent's upward bias. For example, if the principal has a loss function $l(|a - (t-b)|)$ with $b > 0$ while the agent has the same preferences as in the main text, then the agent will be upward biased relative to the principal.[33] Under these assumptions the alignment principle goes through unchanged. One can show that a version of Theorem 1 holds in this new environment. In specific, the same arguments imply that there will be a non-trivial first pooling interval at any optimum. However, this interval of types no longer necessarily pools on an action that is above the ideal point of the lowest type agent $t = 0$, i.e. it is no longer a "floor". A simple example is one in which b is very large so that the optimal action for the principal given the highest type is negative. This would in turn imply that pooling every type on this action is the optimal allocation.

The explicit characterization results in Section 4 depend heavily on quadratic loss. However the recursive approach in Proposition 2 would be valid for any symmetric loss functions described above, as long as Lemma 5 holds.

6.3. Alternative Reputational Preferences

The main analysis is carried out with the assumption that reputational vaue for the agent from a given belief $\mu \in \Delta T$ is proportional to $\mathbb{E}[t|t \sim \mu]$. It can be shown that the analysis extends directly to the case in which this value is proportional to $\mathbb{E}[r(t)|t \sim \mu]$ where r is a continuously differentiable function with derivative bounded away from 0.

As mentioned in the previous subsection, one key feature is maintaining the upward biased preferences of the agent relative to the principal. Given any additively separable and type independent reputational value over beliefs, any allocation will induce a submodular agent loss over actions and types. This means that higher actions will be taken

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[33] In this setting the action space would be augmented to $[-b, \infty)$ in order to include both party's optimal actions for every type.

by higher intervals of types. Thus the agent is upward biased relative to the principal if the reputational value for the prior conditioned on a higher interval is larger than that for a lower interval. Given this point, It seems like this analysis could also extend to "non-expectational" reputational values of the belief as long as they guarantee this associated monotonicity property.

As for the results in Section 4, the exact form of the reputational preference is important. There is only one state variable – the initial loss– in (6) because the reputational loss for a given interval only depends on the interval's length. This memorylessness would not hold for the expectation of a non-linear function of the type.

7. Conclusion

This paper studies whether an agent with a reputational bias should be treated differently than an agent with a material bias. Despite the models having many superficial similarities, I answer this question affirmatively. Specifically, Theorem 1 showed that unlike in material delegation where it is never beneficial to restrict flexibility to low type agents, it is always optimal to impose a floor in reputational delegation. In addition, this is not due to flexibility being sub-optimal per-se. Theorem 2 shows that it is optimal to give full flexibility in the exponential model, when the initial loss is large.

To solve the exponential model, I use the recursive nature of the problem, namely that the first threshold in any continuing allocation pins down the initial loss at that threshold, and thereby the minimized continuation loss for the principal. This method is not specific to the reputational delegation framework, and can be used in other difficult mechanism design problems. The important features are that the allocation over which the principal has commitment (the action) is monotone in the type, and the portion over which the principal does not have commitment (the reputation) is dependent only on the chosen action.

Finally, I assumed throughout the paper that the agent benefits from having a reputation of being a higher type. As discussed, this assumption fits many principal-agent interactions, however it was also partially chosen to make the model directly comparable with the standard material delegation framework in which the agent is biased towards higher actions. It would be interesting for future work to study optimal delegation to agents with different reputational biases. In particular, standard models of expertise, e.g. Ottaviani and Sorensen (2006a), would associate extreme actions with better agents rather than higher actions. Conversely, Bernheim (1994) studies agents with a "preference for conformity", and this reputational bias would push agent's to take more middling actions.

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A. Preliminaries

I first list and prove a series of technical lemmas that will be used in the arguments for the main text results. Lemma 7 shows that it is without loss to restrict to a compact action set. Lemma 8 derives the familiar envelope condition version of incentive compatibility. Lemma 9 shows that the pooling action for a given interval is the minimum action for the endpoint type among all incentive compatible allocations. Lemma 10 uses the previous lemma to derive a lower bound on the loss of the endpoint type for any small interval. Lemma 11 defines continuing allocations in terms of their induced reputation functions. Lemma 12 shows that the set of induced loss functions is compact. Lemma 13 shows that the minimized continuation loss is differentiable. Lemma 14 provides a necessary condition for any optimal pooled interval derived by separating a small set of types near the left endpoint.

For any monotone allocation $x \in IC(T)$ define the endpoint functions $\underline{\tau}, \overline{\tau} : T \to T$ such that $r_x(x(t)) = R(\underline{\tau}(t), \overline{\tau}(t))\ \forall t$. When there is ambiguity, I will notate the allocation in the endpoint functions as $\underline{\tau}_x,\ \overline{\tau}_x$.

It will also be useful to define a sense of a "small interval", such that if this interval is pooled, then the corresponding action is greater than the endpoint type: let $\overline{\delta} > 0$ be defined by $\sqrt{\rho(R(t, t+\delta) - t)} > \delta\ \forall t \in T,\ \forall \delta < \overline{\delta}$. This implies that for $t :\ \overline{\tau}(t) - \underline{\tau}(t) < \overline{\delta}$, it holds that $x(t) > \overline{\tau}(t)$. Since $\lim_{\delta \to 0} \frac{d(R(t,t+\delta))}{d\delta} = 1/2\ \forall t$, such a bound exists.

Lemma 7. *Let $x \in IC(T)$. There exists an allocation $\tilde{x} : T \to [0, B]$ where $B \equiv \sqrt{\rho M + M^2} + M$ such that $\tilde{x} \in IC(T)$ and $L^P(\tilde{x}) \leq L^P(x)$. Let $x \in C(t, u)$. There exists $B_{t,u}$ such that $x(M) < B_{t,u}$.*

Proof. Note that if $x(0) \geq M$, then by monotonicity $x(t) \geq M\ \ \forall t \in T$. This means the principal can improve on x with the allocation $\tilde{x}(t) = R(0, M)\ \forall t \in T$. Thus $x(0) < M$ in any optimal allocation. By incentive compatibility,

$$\begin{aligned} &(x(M) - M)^2 - (x(0) - M)^2 \leq \rho(r^*_x(M) - r^*_x(0)) \\ \implies &x(M) \leq \sqrt{\rho M + M^2} + M \end{aligned}$$

To see the bound for continuing allocations $x \in C(t, u)$, note that $x(t) \in [\sqrt{u} + t, \sqrt{u + \rho(R(t, M) - t)} + t]$ and by IC

$$\begin{aligned} &(x(M) - M)^2 - \rho r^*(M) \leq (x(t) - M)^2 - \rho r^*(t) \\ \implies &(x(M) - M)^2 \leq (x(t) - M)^2, \end{aligned}$$

which because $x(t)$ is bounded gives that $x(M)$ is bounded. *Q.E.D.*

Lemma 8. *$x \in IC(T)$ if and only if x is increasing and*

$$L^A(t|x) = \int_0^t (\rho - 2(x(t') - t'))\,dt' + L^A(0|x)\ \forall t \in T.$$

Moreover, $L^A(t|x)$ has left and right derivatives given by $\rho - 2(x(t^{+(-)}) - t)$ respectively.

Proof. $\forall a \in [0, B]$ from Lemma 7, and $\forall t \in T$ the family of functions for $a \in [0, B]$ given by $L_t^A(a, \cdot|x) = \rho - 2(a - t)$ is uniformly equicontinuous in t. The forward implication thereby follows from theorem 3 in Milgrom and Segal (2002).

To see the reverse implication, take $t_0 < t_1$. The assumption of the lemma gives that

$$\begin{aligned} L^A(t_1|x) - L^A(t_0|x) &= \int_{t_0}^{t_1} (\rho - 2(x(t') - t'))\,dt' \\ \iff (x(t_1) - x(t_0))(x(t_1) + x(t_0) - 2t_0) - \rho(r_x^*(t_1) - r_x^*(t_0)) + 2\int_{t_0}^{t_1} (x(t') - x(t_1))\,dt' &= 0 \\ \implies (x(t_1) - x(t_0))(x(t_1) + x(t_0) - 2t_0) - \rho(r_x^*(t_1) - r_x^*(t_0)) &\geq 0. \end{aligned}$$

The last implication is due to x being increasing. The last line is equivalent to t_0 preferring $x(t_0)$ to $x(t_1)$. The argument for $t_1 < t_0$ is symmetric. *Q.E.D.*

Lemma 9. *Let $y \in C(t, u)$ and $\bar{t} \in \overline{\tau}(T)$ with $t < \bar{t} < t + \bar{\delta}$.*

$$y(\bar{t}) \geq \sqrt{u + \rho(R(t, \bar{t}) - t)} + t.$$

Proof. Suppose the inequality does not hold. Then, since y is increasing, and using the equation for $L_t(t|\cdot)$ in Lemma 8 gives

$$\begin{aligned} &\int_t^{\bar{t}} 2\left(y(t') - \sqrt{u + \rho(R(t, \bar{t}) - t)} - t\right) dt' < 0 \\ \iff L^A(\bar{t}|y) &> \left(\sqrt{u + \rho(R(t, \bar{t}) - t)} - (\bar{t} - t)\right)^2 + \rho(\bar{t} - R(t, \bar{t})). \end{aligned}$$

Note that since $\bar{t} - t < \bar{\delta}$, $y(t') > t'\ \forall t' < \bar{t}$. Thus $y(\bar{t}) < \sqrt{u + \rho(R(t, \bar{t}) - t)} + t$ means that y gives $\bar{t}$ lower material loss than the pooling allocation. However this contradicts the last line of the display, because $r_y^*(\bar{t}) \geq R(t, \bar{t})$. *Q.E.D.*

Lemma 10. *Take any continuing allocation x at (t, u) such that $\overline{\delta} + t > \overline{t} \in \overline{\tau}(T)$.*

$$L^A(\overline{t}|x) > \left(\sqrt{u + \rho(\overline{t} - t)} - (\overline{t} - t)\right)^2.$$

Proof. I start by proving the claim below. The claim says that if the reputations for the agent are higher for any interval, then the agent experiences lower loss. In order to write the claim I will need to notate certain dependences on the distribution. I use subscripts in the agent loss and reputation function to denote dependence on a distribution f – L^A_f and R_f respectively.

Claim 1. *Let $\overline{f}$ strictly monotone likelihood ratio dominate f.*[34] *Let x be a continuing allocation at (t, u) under distribution f with $\overline{\delta} + t \geq \overline{t} \in \overline{\tau}_x(T)$. Let $\overline{x} \in C(t, u)$ under $\overline{f}$ such that $\overline{x}$ and x induce the same interval partition, i.e. $\overline{\tau}_x(t') = \overline{\tau}_{\overline{x}}(t')\ \forall t'$.*[35]

$$L^A_{\overline{f}}(t'|\overline{x}) \leq L^A_f(t'|x)\ \forall t' \in [t, \overline{t}].$$

Proof of Claim: Note that $\forall t_1 < t_2\ R_{\overline{f}}(t_1, t_2) > R_f(t_1, t_2)$ by the definition of MLR dominance. Note that at $t' = t$, $L_{\overline{f}}(t'|\overline{x}) = L_f(t'|x) = u$ by the fact that these are both continuing allocations.

Suppose the claim does not hold, i.e. $\exists t' \in [t, \overline{t}]$: $L^A_{\overline{f}}(t'|\overline{x}) > L^A_f(t'|x)$. Because the reputations are higher under $\overline{x}$, it must be that the material loss is lower under x, i.e. $x(t'^-) < \overline{x}(t'^-)$. By Lemma 8, these losses are both left and right differentiable in t', and

$$\frac{d}{dt^-}\left(L^A_f(t'|x) - L^A_{\overline{f}}(t'|\overline{x})\right) > 0$$
$$\iff x(t'^-) < \overline{x}(t'^-).$$

But this implies that $L^A_f(t''|x) < L^A_{\overline{f}}(t''|\overline{x})\ \forall t'' \in [t, t']$ which contradicts $L^A_{\overline{f}}(t|\overline{x}) = L^A_f(t|x) = u$. This proves the claim

Let $\overline{f}$ be a limiting MLRP dominating distribution such that $R_{\overline{f}}(t_1, t_2) = t_2\ \forall t_1 \leq t_2$. Let $\overline{x}$ be the corresponding continuing allocation at (t, u) under $\overline{f}$ defined by having the same endpoints, i.e. $\overline{\tau}_x(t') = \overline{\tau}_{\overline{x}}(t')\ \forall t'$. Because of the previous claim, $L^A_{\overline{f}}(\overline{t}|\overline{x}) \leq L^A_f(\overline{t}|x)$. Now define an alternative continuing allocation z at (t, u) under distribution $\overline{f}$ that pools types below $\overline{t}$, i.e. it satisfies $z(t') \equiv \sqrt{u + \rho(\overline{t} - t)} + t\ \forall t' < \overline{t}$. Notice that under $\overline{f}$, $r^*_z(\overline{t}) =$

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[34] That is, $\forall t' > t\ \frac{\overline{f}(t')}{f(t')} > \frac{\overline{f}(t)}{f(t)}$.

[35] I prove that such a continuing allocation exists at the beginning of the proof of Lemma 13.

$\bar{t} = r^*_x(\bar{t})$. By Lemma 9, $\bar{t} \leq z(\bar{t}) \leq x(\bar{t})$. Thus $z(t)$ delivers lower distortion and the same reputation so $(\sqrt{\rho(\bar{t} - t) + u} - \bar{t})^2 \leq L^A_f(\bar{t}|\bar{x})$. *Q.E.D.*

Lemma 11. *Take any $t \in T$ and an interval partition I of $[t, M]$ into countable non-degenerate left closed and right open intervals. Define an arbitrary "pooling" subset $P \subset I$ and let the "separating" $S \equiv I \setminus P$. Define $r : [t, M] \to [t, M]$ as*

$$r(\tilde{t}) \equiv \begin{cases} \mathbb{E}[t'|t' \in I_j] & \text{if } \tilde{t} \in I_j \subset P \\ \tilde{t} & \text{if } \tilde{t} \in I_j \subset S \end{cases}.$$

Let $\mathcal{R}_t$ be the set of such reputation functions. For every $u \geq 0$, $t \in T$ and every $r \in \mathcal{R}_t$, there exists a unique continuing allocation $x_{r,u,t} \in C(t, u)$ such that $r^*_{x_{r,u,t}}(t') = r(t)$.

Proof. I construct such an allocation iteratively. let $\overline{\tau}, \underline{\tau}$ be the associated threshold functions for r, i.e., $r(\tilde{t}) = \mathbb{E}[t'|t' \in [\underline{\tau}(\tilde{t}, \overline{\tau}(\tilde{t})]$. Suppose an allocation x is monotone and satisfies

$$(x(s) - s)^2 + \rho(s - r(s)) = \min_{t' \in T}(x(t') - s)^2 + \rho(s - r(t')).$$

for $s \in [t, \bar{t}]$. Because the reputations correspond to an interval partition, one can extend $x(t') \equiv x(\bar{t})$ and still satisfy the above equation and monotonicity for $s \in [t, \overline{\tau}(\bar{t})]$. So, without loss, take a threshold as $\bar{t}$, i.e., $\bar{t} = \underline{\tau}(\bar{t})$. Now consider the case in which the first threshold $\tilde{t} \in \overline{\tau}([\bar{t}, M])$ of the interval partition greater than $\bar{t}$ exists and is greater than $\bar{\delta} + \bar{t}$. In this case define $x(t') \equiv \sqrt{L^A(\bar{t}|x) + \rho(r(\bar{t}) - \bar{t})} + \bar{t}\ \forall t' \in [\bar{t}, \tilde{t}]$ which is defined because $L^A(\bar{t}|x) \geq 0$. If instead, one cannot find such a threshold, then define $\hat{t}$ to be the highest threshold in $(\bar{t}, \bar{t} + \bar{\delta}]$. Define $x(t') \equiv \sqrt{l(t') + \rho(r(t') - t')} + t'$, where $l(t')$ solves the differential equation $l'(t') = \rho - 2\sqrt{l(t') - \rho(t - r(t'))}$ on $[\bar{t}, \hat{t}]$ with initial condition $l(\bar{t}) \equiv L^A(\bar{t}|x)$. A solution exists by theorem 1 in Persson (1975). Repeating this process once more guarantees existence of x for $s \in [t, \tilde{t} + \bar{\delta}]$. And so we can repeat this process a finite number of times and extend to $[t, M]$. By construction, $l(s)$ is absolutely continuous with right and left derivatives defined as $l'(s^{+(-)}) = \rho - 2(x(s^{+(-)}) - s))$. By construction $r^*_x(s) = r(s)$. Next note that monotonicity of $x(s)$ follows the monotonicity of $r(s)$: (i) on pooling regions, the construction gives that $x(s)$ is constant, (ii) on separating regions where $r(s) = s$, the construction gives that $x'(s) = \frac{\rho}{2(x(s)-s)} > 0$, and (iii) at threshold types a discontinuous jump up in r propagates to x. Thus, this construction means that the allocation satisfies the envelope formula in Lemma 8, and it is an IC allocation.

I will next show that r uniquely pins down the continuing allocation. Suppose not, i.e. $\exists x, y \in C(t, u)$ such that $r^*_x = r^*_y$ but $x \neq y$. Take $s \equiv \inf\{t' : L^A(t'|x) \neq L^A(t'|y)\}$. Since the agent's loss is continuous in his type for any IC allocation and $L^A(t|x) = L^A(t|y) = u$, it must be that $L^A(s|x) = L^A(s|y)$. Suppose first that there exists $t' < s + \overline{\delta}$ such that $L^A(t'|x) \neq L^A(t'|y)$ and $\underline{\tau}(t') \leq s$. Let $\underline{\tau}(s) \equiv \underline{s}$. Because the reputations are the same, the loss being different means that $x(t') \neq y(t')$. But since $\underline{\tau}(t') \leq s$, $x(t') = x(\underline{s}) \neq y(\underline{s}) = y(t')$. But this contradicts that because $L^A(\underline{s}|x) = L^A(\underline{s})|y)$,

$$x(\underline{s}) = \sqrt{L^A(\underline{s}|x) + \rho(r(\underline{s}) - \underline{s})} + \underline{s} = \sqrt{L^A(\underline{s}|y) + \rho(r(\underline{s}) - \underline{s})} + \underline{s} = y(\underline{s}).$$

Take $\tilde{t}' \in [s, s + \overline{\delta})$. Since $\underline{\tau}(\tilde{t}) > s$, by definition of s, I can take $t' \in (s, \underline{\tau}\tilde{t})$ so that $s < \underline{\tau}(t') \leq \overline{\tau}(t') < s + \overline{\delta}$ such that either $L^A(t'|x) > L^A(t'|y)$ or $L^A(t'|x) < L^A(t'|y)$. I will deal with the first inequality with the opposite case being symmetric. Because agent losses are continuous in the type, there exists $\underline{s} \geq s$ such that $L^A(t''|x) > L^A(t''|y)$ $\forall t'' \in (\underline{s}, t']$ and $L^A(\underline{s}|x) = L^A(\underline{s}|y)$. By definition of $\overline{\delta}$, this means that $x(t'') > y(t'') > t''$ $\forall t'' \in (\underline{s}, t']$. But this is a contradiction, because by Lemma 8,

$$\begin{aligned} & L^A(t'|x) - L^A(t'|y) \\ = & 2\int_{\underline{s}}^{t'} \left(y(t'') - x(t'')\right) dt'' + L^A(\underline{s}|x) - L^A(\underline{s}|y) \\ < & 0. \end{aligned}$$

Q.E.D.

Lemma 12. *Take the set of loss functions induced by IC allocations and continuing allocations to be* $\mathcal{L} \equiv \{L^A(\cdot|x) : x \in IC(T)\}$ *and* $\mathcal{L}_{t,u} \equiv \{L^A(\cdot|x) : x \in C(t,u)\}$. *Both* $\mathcal{L}$ *and* $\mathcal{L}_{t,u}$ *are compact.*

Proof. Note that by Lemma 7, it is without loss to take IC allocations $x : T \to [0, B]$. By the envelope theorem, $L^A_t(t|x) = \rho - 2(x(t) - t)$. Notice that this derivative is uniformly bounded because $(x(t), t)$ are from a compact set. Take the set of loss functions induced by IC allocations to be $\mathcal{L} \equiv \{L^A(t|x) : x \in IC(T)\}$. $\mathcal{L}$ is a set of uniformly equicontinuous functions and is thereby relatively compact by the Arzela – Ascoli theorem. In order to show that $\mathcal{L}$ is compact I only need to show it is closed.

Let x_n be a sequence of IC allocations with associated agent loss functions $L^A(t|x_n)$, associated reputation functions r^*_n and r_n, and associated threshold functions $\underline{\tau}_n$ and $\overline{\tau}_n$. Suppose $L^A(t|x_n)$ converges, and by the above argument this convergence is uniform across $t \in T$. Define $L_n(t) \equiv L^A(t|x_n) - \rho t + t^2$ and $L^*(t) \equiv \lim_{n\to\infty} L_n(t)$ to be the uniform limit. Recall that by convention each $x_n(t)$ is right continuous. Note that by Lemma 8, for $\bar{t} > \underline{t}$, $L_n(\bar{t}) - L_n(\underline{t}) = -2\int_{\underline{t}}^{\bar{t}} x_n(t)$, so each L_n is concave because each x_n is increasing by incentive compatibility. Therefore L^* is also concave and admits right continuous right derivatives. Accordingly label these right derivatives as $x^*(t) \equiv (-1/2)\frac{d(L^*(t)^+)}{dt}$. Let $r^*(t) \equiv \mathbb{E}[t'|x^*(t') = t]$. Since L^* is concave this expectation is taken over an interval of types and x^* is a monotonic allocation. Let $\underline{\tau}^*$ and $\overline{\tau}^*$ be the corresponding threshold functions. Note that because L_n converges uniformly to L^*, for any length δ interval, this gives

$$\forall \delta, \varepsilon > 0,\ \exists N :\ \forall n > N,\ \tilde{t} \in T, \quad \left| \frac{\int_{\tilde{t}}^{\tilde{t}+\delta} x_n(t)dt}{\delta} - \frac{\int_{\tilde{t}}^{\tilde{t}+\delta} x^*(t)dt}{\delta} \right| < \varepsilon, \tag{10}$$

i.e., the average actions on any interval become arbitrarily close as n becomes large. Notice that (10) also implies that

$$\forall \delta, \varepsilon > 0,\ \forall \tilde{t} \in T,\ \ \exists N :\ \forall n > N, \quad \min_{t' \in [\tilde{t}, \tilde{t}+\delta]} |x_n(t') - x^*(t')| < \varepsilon. \tag{11}$$

To see this implication note that if it were not true then there exists (abusing notation) a subsequence such that $|x_n(t_n) - x^*(t_n)| > \varepsilon\ \forall n,\ \forall t' \in [t, t+\delta]$. Since x^* is right continuous, take $\hat{\delta}$ such that $x^*(t') - x^*(t) < \varepsilon/2\ \forall t' \in [t, t+\hat{\delta}]$. Because $|x_n(t') - x^*(t')| > \varepsilon\ \forall t' \in [t, t+\delta]$ by hypothesis, the sets $\{t' \in [t, t+\delta] :\ x_n(t') < x^*(t')\}$ and $\{t' \in [t, t+\delta] :\ x_n(t') > x^*(t')\}$ are both intervals given by $[t, t_n)$ and $[t_n, t+\delta]$ (possibly empty) respectively. Take a further subsequence such that (i) $\forall n\ t_n - t \geq \hat{\delta}/2$ or (ii) $\forall n\ t + \hat{\delta} - t_n \geq \hat{\delta}/2$, at least one of which must exist. Arguing in case (i), as case (ii) is symmetric,

$$\begin{aligned} &\left| \frac{\int_t^{t+\hat{\delta}/2} x_n(\tilde{t})d\tilde{t}}{\hat{\delta}/2} - \frac{\int_t^{t+\hat{\delta}/2} x^*(\tilde{t})d\tilde{t}}{\hat{\delta}/2} \right| \\ =&\frac{\int_t^{t+\hat{\delta}/2} (x^*(\tilde{t}) - x_n(\tilde{t}))d\tilde{t}}{\hat{\delta}/2} \\ >&\varepsilon. \end{aligned}$$

where the last inequality is by hypothesis. But using ε and $d = \delta/2$ in (10) with $\tilde{t} = t$ gives the opposite inequality as in the display for large enough n, a contradiction.

I will use these facts to prove the following claim.

Claim 2. *$\forall t, s \in T\ \forall \varepsilon > 0$, $\exists t_n \to t$ such that $\forall n \geq N$ $|L^A(x^*(t), s|x^*) - L^A(x_n(t_n), s|x_n)| < \varepsilon$*

Note that Claim 2 proves incentive compatibility of x^* through incentive compatibility of each x_n. To see this suppose x^* is not incentive compatible so there exists $\tilde{t}, \tilde{s}$ and $\tilde{\varepsilon} > 0$ such that $L^A(x^*(\tilde{t}), \tilde{s}|x^*) < L^A(x^*(\tilde{s}), \tilde{s}|x^*) + \tilde{\varepsilon}$. Using Claim 2 with $\varepsilon = \tilde{\varepsilon}/3$ twice for $(t, s) = (\tilde{t}, \tilde{s})$

to get a $t_n \to t$ sequence and for $(t, s) = (\tilde{s}, \tilde{s})$ to get an $s_n \to s$ sequence such that for sufficiently large n, s strictly prefers mimicking t_n to s_n under x_n by at least $\varepsilon/3$. But then because $\forall t$, $L_t^A(t|x_n) = \rho - 2(x_n(t) - t) \in [\rho - 2B, \rho + 2M]$, and $s_n \to s$, for sufficiently large n $|L^A(s_n, s)|x_n) - L^A(s|x_n)| < \varepsilon/3|$ which means that s strictly prefers to deviate to t under x_n.

Proof of Claim:

Case 1: x^* is strictly increasing to the right at t . This means that $r^*(t) = t$. I will show first that

$$\forall \varepsilon > 0 \; \exists \; \underline{\eta} < \overline{\eta} < \varepsilon \text{ and } N \text{ such that } \forall \tilde{s} \in [t + \underline{\eta}, t + \overline{\eta}] \text{ and } \forall n \geq N \; |r_n^*(\tilde{s}) - t| < \varepsilon \tag{12}$$

Because the reputation function is continuous with respect to the boundaries of the interval, $r_n^*(\tilde{t}) - t > \varepsilon$ implies that $\overline{\tau}_n(\tilde{t}) > \tilde{\varepsilon} + t$ for some $\tilde{\varepsilon} > 0$. Since x^* is right continuous by definition and strictly increasing at t by assumption there exists t_1, t_2 such that $t < t_1 < t_2 < \tilde{\varepsilon}$ and $x^*(t) < x^*(t_1) < x^*(t_2)$. Now suppose toward a contradiction that $\exists \; s_n < (t_1 + t)/2$ such that $r_n^*(s_n) - t > \varepsilon$. Thus, the fact that $\overline{\tau}_n(s_n) > \tilde{\varepsilon} + t$ means that $x_n(t') \equiv y_n$ is constant, $\forall t' \in ((t + t_1)/2, t + \tilde{\varepsilon})$.

There are three cases depending on whether there exists a subsequence such that (i) $y_n \in [x^*(t_1), x^*(t_2)]$, (ii) $y_n < x^*(t_1)$, or (iii) $y_n > x^*(t_2)$. In case (i) using (11) twice with $\delta = \min\{(t_1 - t)/2, (\tilde{\varepsilon} - t_2)/2\}$ and $\varepsilon = (x^*(t_2) - x^*(t_1))/2$ in for $\tilde{t} = (t + t_1)/2$ and $\tilde{t} = t_2$ gives that for sufficiently high n, respectively $|y_n - x^*(t_1)| < \varepsilon$ and $|y_n - x^*(t_2)| < \varepsilon$ which contradicts the definition of ε, i.e., y_n cannot be arbitrarily close to both $x^*(t_1)$ and $x^*(t_2)$ The argument for cases (ii) and (iii) are symmetric, so I only argue for a contradiction in the former. In case (ii), using $\tilde{\delta} = (\varepsilon - t_2)/2$ and and $\varepsilon = (x^*(t_2) - x^*(t_1))$ for $\tilde{t} = t_2$ gives that for sufficiently high n, $|y_n - x^*(t_2)| < \varepsilon$ which contradicts the definition of ε and the hypothesis that $y_n < x^*(t_1)$. This means that $\forall t' \in [0, (t_1 + t)/2)$ $r_n(s_n) - t < \varepsilon$.

Now take $\overline{\eta} \equiv (t_1 - t)/2$, and take arbitrary $\underline{\eta}$ such that $0 < \underline{\eta} < \overline{\eta}$. Thus, in order for (12) to be violated there exists a sequence $s_n \in (t + \underline{\eta}, t + \overline{\eta})$ such that $\forall n$ $|r_n^*(s_n) - t| > \varepsilon$, which by the above argument implies it must be that $\forall n$, $t - r_n(s_n) > \varepsilon$. This implies that $\forall n$ $\underline{\tau}_n(s_n) < t$ and $x_n(t') \equiv y_n$ is constant for all large n in the subsequence, and $t' \in [t, t + \underline{\eta}]$. Now, again because $x^*(t)$ is strictly increasing and right continuous at at t by definition, there exists s', s'' such that $t < s' < s'' < t + \underline{\eta}$ such that $x^*(t) < x^*(s') < x^*(s'')$. There are again three cases depending on whether there exists a subsequence such that (i) $y_n \in [x^*(s'), x^*(s'')]$, (ii) $y_n < x^*(s')$, or (iii) $y_n > x^*(s'')$, and the argument for a contradiction is identical to that at the end of the previous paragraph. This proves (12).

By right continuity of x^*, for any ε_0 there exists small δ_0 such that $x^*(t') - x^*(t) < \varepsilon_0$ $\forall t' \in [t, t+\delta_0]$. By (12), one can take $0 < \underline{\eta} < \overline{\eta} < \delta_0$ such that $\forall \tilde{s} \in [t+\underline{\eta}, t+\overline{\eta}]$ and $\forall n \geq N$ $|r_n^*(\tilde{s}) - t| < \varepsilon$. In addition, using $\delta = \overline{\eta} - \underline{\eta}$ and $\varepsilon = \varepsilon_0$ in (11) for $\tilde{t} = t + \underline{\eta}$ gives that for sufficiently large $n > N$ there exists $t_n \in [t+\underline{\eta}, t+\overline{\eta}]$ such that $|x_n(t_n) - x^*(t_n)| < \varepsilon_0$. This means that $|x_n(t_n) - x^*(t)| < 2\varepsilon_0$. in turn, $L^A(x_n(t_n), s|x_n) = (x_n(t_n) - s)^2 + \rho(s - r_n(t_n))$ becomes arbitrarily close to $(x^*(t) - s)^2 + \rho(s - r^*(t)) = L^A(x^*(t), t|x^*)$ as we takes $\varepsilon_0 \to 0$. Moreover, as $\varepsilon_0 \to 0$, it holds that $\delta_0 \to 0$ and so $t_n \to t$. This proves the claim in this case.

Case 2: $x^*(t) \equiv \tilde{x}$ is constant at t, i.e., $\underline{\tau}^*(t) \equiv \underline{t} \leq t < \bar{t} \equiv \overline{\tau}^*(t)$. First I will show that

$$\forall \varepsilon > 0,\ 0 < \eta < (\bar{t} - \underline{t})/2,\ \exists N:\quad \forall n \geq N,\ x_n(t') = y_n \text{ is constant for } t' \in [\underline{t} + \eta, \bar{t} - \eta]. \tag{13}$$

Notice and $|y_n - \tilde{x}| < \varepsilon$. Notice that if the (13) holds then from (10), I can take a further N such that $\forall n \geq N$, $|y_n - \tilde{x}| < \varepsilon$.

Towards a contradiction suppose that there exists a subsequence such that x_n is not constant on $t' \in [\underline{t} + \eta, \bar{t} - \eta]$, i.e., $\underline{x}_n \equiv x_n(\underline{t} + \eta) < x_n(\bar{t} - \eta) \equiv \overline{x}_n$. Denote $\overline{\tau}_n(\underline{t} + \eta) \equiv \tilde{t}_n \in [\underline{t} + \eta, \bar{t} - \eta]$, and note that by incentive compatibility

$$\begin{aligned}
&(\overline{x}_n - \tilde{t}_n)^2 - (\underline{x}_n - \tilde{t}_n)^2 \geq \rho(r_n(\overline{x}) - r_n(\underline{x})) \\
\iff &(\overline{x}_n - \underline{x}_n) \geq \frac{\rho(r_n(\overline{x}) - r_n(\underline{x}))}{\overline{x}_n + \underline{x}_n - 2\tilde{t}_n} \\
\implies &(\overline{x}_n - \underline{x}_n) \geq \frac{\min_{t' \in [\underline{t}+\eta, \bar{t}-\eta]} \rho(R(t', \bar{t} - \eta) - R(\underline{t} + \eta, t'))}{2B} \equiv \tilde{\varepsilon} > 0.
\end{aligned}$$

There are three sub-cases. First suppose that there exists a subsequence such that $\underline{x}_n \leq \tilde{x} \leq \overline{x}_n$. Take $\varepsilon = \tilde{\varepsilon}/2$ and $\delta = \eta$ in (10) twice with $\tilde{t} = \underline{t}$ and $\tilde{t} = \bar{t} - \delta$ respectively combined with monotonicity of x_n to get both $|\tilde{x} - \underline{x}_n| < \tilde{\varepsilon}/2$ and $|\tilde{x} - \overline{x}_n| < \tilde{\varepsilon}/2$ which contradicts the last line of the display above. Second suppose that it is not the first sub-case so either there exists (i) a subsequence such that $\underline{x}_n > \tilde{x}$ or (ii) a subsequence such that $\overline{x}_n < \tilde{x}$. The argument for a contradiction is symmetric in each of these subcases so consider that (i) holds. In this case, by the last line of the display above, $\overline{x}_n - \tilde{x} > \tilde{\varepsilon}$. But taking $\varepsilon = \tilde{\varepsilon}/2$, $\delta = \eta$, $\tilde{t} = \bar{t} - \eta$ in (10) along with monotonicity gives that $|\tilde{x} - \overline{x}_n| < \tilde{\varepsilon}/2$, a contradiction. This proves that (13) holds.

Next I will show that $\forall \eta \in (0, (\bar{t} - \underline{t})/2)$, $\exists N$ such that $\forall n > N$, (i) $\underline{\tau}_n(\underline{t} + \eta) > \underline{t} - \eta$ and (ii) $\overline{\tau}_n(\bar{t} - \eta) < \bar{t} + \eta$. I will prove (i) as the argument for (ii) is symmetric. Note that from (13), x_n is constant on $[\underline{t} + \eta, \bar{t} - \eta]$ at y_n for large enough n. Towards a contradiction suppose that there exists a subsequence such that $x_n(t') = y_n$ $\forall t' \in [\underline{t} - \eta, \underline{t} + \eta]$. But since $\underline{\tau}^*(t) = \underline{t}$ by definition, $x^*(\underline{t} - \eta/2) \equiv \underline{x} < \tilde{x} - \tilde{\varepsilon}$ for some $\tilde{\varepsilon} > 0$. By taking $\delta = \eta$, $\varepsilon = \tilde{\varepsilon}/2$, and $\tilde{t} = \underline{t}$ in (10) this gives that $|\tilde{x} - y_n| < \tilde{\varepsilon}/2$. But then using (10) again with $\tilde{t} = \underline{t} - \eta$, $\varepsilon = \tilde{\varepsilon}/2$, and $\delta = \eta/2$ along with monotonicity of x^* gives $|y_n - \underline{x}| < \tilde{\varepsilon}/2$. These two inequalities contradict the definition of $\tilde{\varepsilon}$.

These points imply that for any $\eta \in (0, (\bar{t} - \underline{t})/2)$, there exists large n such that $r_n^*(t+\eta) = R(\underline{t}_n, \bar{t}_n)$ for some $\underline{t}_n \in [\underline{t} - \delta, \underline{t} + \delta]$ and $\bar{t}_n \in [\bar{t} - \eta, \bar{t} + \eta]$. Since the reputation function $R(\cdot, \cdot)$ is continuous in both arguments, and $r^*(t) = R(\underline{t}, \bar{t})$, $r_n^*(\underline{t} + \eta)$ becomes arbitrarily close to $r^*(t)$ as $\eta \to 0$. In addition, I have also demonstrated that for any small $\varepsilon > 0$, for large enough n, $|x_n(t+\delta) - x^*(t)| = |y_n - \tilde{x}| < \varepsilon$. This means that $L^A(x_n(t+\delta), s|x_n)$ can be made arbitrarily close to $L^A(x^*(t), s|x^*)$ proving the claim.

Thus $\mathcal{L}$ is compact. To see that $\mathcal{L}_{t,u}$ is compact, notice that the same arguments above imply that $\mathcal{L}_{t,u}$ is relatively compact, and that a sequence $\{L_n\} \in \mathcal{L}_{t,u}$ converges to an L^* in $\mathcal{L}_{t,u'}$ for some $u' \geq 0$. But note that $u' = \lim_{n\to\infty} L_n(t) = u$ because by definition $L_n(t) = u \; \forall n$. Thus $\mathcal{L}_{t,u}$ is closed and so it is compact as well. *Q.E.D.*

Lemma 13. *$\forall t \in T$ and $u > 0$, $V(t,u)$ has left and right derivatives in u denoted $V_{u^-}(t,u)$ and $V_{u^+}(t,u)$ respectively.*

Proof. In order to show differentiability of V, I will redefine the set of continuing allocations by their induced reputations. Thus, for each $r \in \mathcal{R}_{\sqcup}$ recall $x_{r,u,t}$ is the unique continuing allocation in $C(t,u)$ with reputation function $r^*_{x_{r,u,t}}(t') = r(t') \; \forall t' \in [t, M]$ by Lemma 11. I can rewrite the problem in (3) as,

$$V(t,u) = \frac{1}{F([t,M])} \min_{r \in \mathcal{R}} \int_t^M L^A(t'|x_{r,u,t}) f(t') dt'$$

Claim 3. *Fix a $u_1 > 0$. There exists a constant $\lambda(u_1) > 0$ such that for every $u \geq u_1$ and continuing allocation $x \in C(t,u)$, every threshold $\tau \in \tau(T)$ has $L^A(\tau \mid x) \geq \lambda(u_1)$.*

Proof of Claim: Let $\delta_1 = \min\{\bar{\delta}, \rho/2\}$. First I show that for any allocation x with $s_1, s_2 \in \tau(T)$ and $s_1 + \delta_1 > s_2 > s_1$, it holds that $L^A(t|x) < \rho^2/16 \implies L^A(s_2|x) \geq L^A(s_1|x)$. To see this denote $\eta \equiv s_2 - s_1$ and $u \equiv L^A(s_1|x)$. Note that by Lemma 10 $L^A(s_2|x) > (\sqrt{u + \rho\eta} - \eta)^2$. Thus the inequality holds if

$$\begin{aligned}
&(\sqrt{u + \rho\eta} - \eta)^2 > u \\
\iff &\rho - 2\sqrt{u + \rho\eta} + \eta > 0 \\
\impliedby &\rho - 2\sqrt{\rho^2/16 + \rho\eta} + \eta \geq 0 \\
\iff &1 - 2\sqrt{1/16 + \eta/\rho} + \eta/\rho \geq 0
\end{aligned}$$

Notice that the last LHS only depends on one parameter $\tilde{\eta} \equiv \eta/\rho$. The LHS is continuous, strictly positive at $\tilde{\eta} = 0$, and has two roots given by $\{1/2, 3/2\}$, and so it is strictly positive for $\tilde{\eta} < 1/2$ or $\eta < \rho/2$.

Next I show there exists $\delta_2 > 0$ such that for every allocation x with $s_1, s_2 \in \tau(T)$ and $s_1 + \delta_2 > s_2 > s_1$, it holds that $L^A(s_1|x) \geq \rho^2/16 \implies L^A(s_2|x) \geq \rho^2/32$. To see this, again denote $\eta \equiv s_2 - s_1$ and $u \equiv L^A(s_1|x)$, and note that by Lemma 10 that

$$L^A(s|x) > (\sqrt{u + \rho\eta} - \eta)^2.$$

For small η this RHS is increasing in u, so it is sufficient to prove that for small η,

$$(\sqrt{\rho^2/16 + \rho\eta} - \eta)^2 > \rho^2/32.$$

But this holds at $\eta = 0$ and the LHS is continuous in η so the result holds.

Now define $\tilde{\delta} \equiv \min\{\delta_1, \delta_2\}$ and let $\underline{R} \equiv \min_{\{(t_1,t_2):t_2 \geq t_1 + \tilde{\delta}\}} t_2 - R(t_1, t_2)$. Note that $\underline{R} > 0$ because the density is bounded away from zero and from above. Take $u_1 > 0$ and define $\lambda(u_1) \equiv \min\{u_1 - \varepsilon, \rho^2/32, \rho\underline{R}\}$ for small $\varepsilon < u_1$. Now consider any continuing allocation $x \in C(t, u)$ such that $u \geq u_1$ and suppose that the set of thresholds $\{t' \in \tau(T) : L^A(t'|x) \leq \lambda(u_1)\}$ is nonempty. Note that because the agent loss is continuous in t and the set of thresholds is closed, $\underline{t} \equiv \min\{t' \in \tau(T) : L^A(t'|x) \leq \lambda(u_1)\}$ exists and has $L^A(\underline{t}|x) \leq \lambda(u_1)$. Since x is a continuing allocation, $L^A(t|x) = u \geq u_1 > \lambda(u_1)$ so $\underline{t} > t$. Consider there is a threshold $\tilde{t} \in \tau(T) \cap (\underline{t} - \tilde{\delta}, \underline{t})$. If $L^A(\tilde{t}) \geq \rho^2/16$, then $L^A(\underline{t}) > \rho^2/32 > \lambda(u_1)$ by the definition of δ_2 which is a contradiction. If $L^A(\tilde{t}|x) < \rho^2/16$, then by the definition of δ_1, $\lambda(u_1) \geq L^A(\underline{t}|x) > L^A(\tilde{t}|x)$ which contradicts the definition of $\underline{t}$. Now suppose there does not exist a threshold $\tilde{t} \in \tau(T) \cap (\underline{t} - \tilde{\delta}, \underline{t})$. Then take $\tilde{t} \equiv \tau(\underline{t}^-) \leq \underline{t} - \tilde{\delta}$ to be the first threshold below $\underline{t}$. But then $L^A(\underline{t}|x) \geq \rho(\underline{t} - r_x^*(\underline{t})) = \rho(\underline{t} - R(\tilde{t}, \underline{t})) \geq \rho\underline{R}$. But this is a contradiction as well, proving the claim.

Claim 4. *Fix $0 < u_1 \leq u \leq u_2$ and let $\lambda(u_1)$ be as in Claim 3. Define $\tilde{\delta} \equiv \min\{\lambda(u_1)/(2\rho), \overline{\delta}\}$. Take an arbitrary reputation function r on $[t, M]$, and let $\{I_j\}_{j=1,..,J}$ be the set of pooling intervals longer than $\tilde{\delta}$ with $J \leq M/\tilde{\delta}$ where $I_j \equiv [a_j, b_j]$. Let $K = [t, M] \setminus \cup_{j=1}^{J} I_j$ and $K_j \equiv [\alpha_j, \beta_j]$ be its finite connected components for $j = 0, 1..., J$. Define $L(s, u; r) \equiv L^A(s|x_{r,u,t})$. Let $\overline{R} := \min_t \big(R(t, t + \tilde{\delta}) - t\big) > 0$ be the minimum additional reputation such an interval carries over its initial threshold. The derivative $L_u(s, u; r)$ exists and is uniformly equicontinuous in u. and is recursively defined by* *(i) For $s \in K_j$*

$$L_u(s,u;r) = L_u(\alpha_j,u;r)\exp\left(-\int_{\alpha_j}^{s}\frac{dv}{x_{r,u,t}(v)-v}\right),$$

(ii) For $s \in I_j$

$$L_u(s,u;r) = L_u(a_j,u;r)\frac{x_u^j - s}{x_u^j - a_j},$$

where $x_u^j = x_{r,u,t}(s)$ denotes the pooled action on I_j.

(iii) At the initial type t, $L_u(t,u;r) = 1$

Finally, for $w \equiv \max\left\{1, \frac{M}{\sqrt{\rho \overline{R}}}\right\}$, it holds that $\forall s \in [t,M]$, that $|L_u(s,u;r)| \leq w^J$.

Proof of Claim: For $s \in [t,M]$ and small $h \neq 0$ define

$$Q_h(s) \equiv \frac{L(s,u+h;r) - L(s,u;r)}{h} \quad \text{and}$$
$$\kappa_h(s) \equiv \frac{2}{\sqrt{L(s,u+h;r) - \rho(s-r(s))} + \sqrt{L(s,u;r) - \rho(s-r(s))}}.$$

First take $s \in K_j = [\alpha_j, \beta_j]$. By construction, $x_{r,u,t}(s) - s \geq 0$ and so

$$x_{r,u,t}(s) - s = \sqrt{L(s,u;r) - \rho(s-r(s))}.$$

Hence by the Lemma 8

$$\begin{aligned}
L(s,u;r) =& L(\alpha_j,u;r) + \int_{\alpha_j}^{s}\Big(\rho - 2(x_{r,u,t}(v) - v)\Big)\,dv \\
=& L(\alpha_j,u;r) + \int_{\alpha_j}^{s}\Big(\rho - 2\sqrt{L(v,u;r) - \rho(v-r(v))}\Big)\,dv
\end{aligned}$$

Subtracting the two integral equations and dividing by h gives

$$Q_h(s) = Q_h(\alpha_j) - \int_{\alpha_j}^{s}\kappa_h(v)Q_h(v)\,dv,$$

The solution to this integral equation is given by

$$Q_h(s) = Q_h(\alpha_j)\exp\left(-\int_{\alpha_j}^{s}\kappa_h(v)\,dv\right).$$

This implies that $|Q_h(s)| \leq |Q_h(\alpha_j)|$, i.e., Q_h is decreasing in magnitude on the connected components of K.

Now consider $s \in I_j = [a_j, b_j]$. Since I_j is a pooling interval, $x_{r,u,t}(v) = x_u^j \;\forall v \in I_j$. Let r_j denote the common reputation on I_j. To satisfy IC for type a_j, $x_u^j = a_j + \sqrt{L(a_j, u; r) + \rho(r_j - a_j)}$. Thus $L(s, u; r) = (x_u^j - s)^2 + \rho(s - r_j)$. Take $h > 0$ and note that

$$
\begin{aligned}
&L(s, u+h; r) - L(s, u; r) \\
=&L(a_j, u+h; r) - L(a_j, u; r) - 2(s - a_j)\left(\sqrt{L(a_j, u+h; r) + \rho(r_j - a_j)} - \sqrt{L(a_j, u; r) + \rho(r_j - a_j)}\right) \\
=&\Big(L(a_j, u+h; r) - L(a_j, u; r)\Big)\left(1 - 2\frac{(s - a_j)}{\sqrt{L(a_j, u+h; r) + \rho(r_j - a_j)} + \sqrt{L(a_j, u; r) + \rho(r_j - a_j)}}\right).
\end{aligned}
$$

Dividing by h gives

$$
Q_h(s) = Q_h(a_j)\left(1 - (s - a_j)\kappa_h(a_j)\right)
$$

Since $r_j - a_j > \overline{R}$ by construction, $\kappa_h(a_j) < \frac{1}{\sqrt{\rho \overline{R}}}$ and $|Q_h(s)| < |Q_h(a_j)| \max\left\{1, \frac{M}{\sqrt{\rho \overline{R}}}\right\}$. Let $w \equiv \max\left\{1, \frac{M}{\sqrt{\rho \overline{R}}}\right\}$. This bound, the fact established above that Q_h is decreasing in magnitude on connected components of K, that there are at most J I_j "long" pooling intervals, and finally that at the initial point, because $L(t, u, ; r) = u$ for any continuing allocation, $Q_h(t) = 1$, gives recursively that $|Q_h(s)| < w^J$. That is $L(s, u; r)$ is uniformly Lipschitz in u across all r, u and $s \in [t, M]$, and thereby also continuous. This means we can evaluate the limit of $\kappa_h(v)$ as $h \to 0$ as $\frac{1}{x_{r,u,t}(v) - v}$. Correspondingly evaluate the limits of the formulas for Q_h on each type of interval as $h \to 0$ to get

$$
L_u(s, u; r) = \begin{cases} L_u(\alpha_j, u; r)\exp\left(-\int_{\alpha_j}^{s} \frac{dv}{x_{r,u,t}(v) - v}\right) & s \in K_j \\ L_u(a_j, u; r)\frac{x_u^j - s}{x_u^j - a_j} & s \in I_j \end{cases}.
$$

Take any $s \in K_j$. Note that by construction $\overline{\tau}(s) - \underline{\tau}(s) \leq \overline{\delta}$ and so $x_{r,u,t}(\overline{\tau}(s)) > \overline{\tau}(s)$ so $x_{r,u,t}(s) - s$ is decreasing on $[\underline{\tau}(s), \overline{\tau}(s)]$. But by Claim 3

$$
L(\overline{\tau}(s), u; r) = (x_{r,u,t}(\overline{\tau}(s)) - \overline{\tau}(s))^2 + \rho(\overline{\tau}(s) - r(\overline{\tau}(s))) > \lambda(u_1).
$$

But since $\overline{\tau}(s) - \underline{\tau}(s) < \tilde{\delta}$, it holds that $\rho(\overline{\tau}(s) - r(\overline{\tau}(s))) \leq \rho\tilde{\delta} \leq \lambda(u_1)/2$, and so $x_{r,u,t}(\overline{\tau}(s)) - \overline{\tau}(s) > \sqrt{\lambda(u_1)/2}$ and $x_{r,u,t}(s) - s > \sqrt{\lambda(u_1)/2}$.

Now note that because $I_j = [a_j, b_j]$ is a pooling interval longer than $\tilde{\tilde{\delta}}$,

$$x_u^j - a_j = \sqrt{L(a_j, u; r) + \rho(R(a_j, b_j) - a_j)} > \sqrt{\rho z \tilde{\tilde{\delta}}}.$$

Define $\tilde{c} \equiv \min\left\{\sqrt{\lambda(u_1)/2}, \sqrt{\rho z \tilde{\tilde{\delta}}}\right\}$

Now consider $s \in [\alpha_j, \beta_j]$ so that

$$L_u(s, u; r) = L_u(\alpha_j, u; r) \exp\left(-\int_{\alpha_j}^{s} \frac{dv}{x_{r,u,t}(v) - v}\right),$$

The previous argument gives that $x_{r,u,t}(s) - s > \tilde{c} > 0$ so

$$x_{r,u,t}(s) = \sqrt{L(s, u; r) + \rho(r(s) - s)} + s.$$

Note that because (i) $L(s, u; r) \geq \lambda(u_1) > 0$ for $s \in K$ and $u \geq u_1$ as previously proved, (ii) the mapping $l \mapsto \sqrt{l + \rho(r(s) - s)} + s$ is uniformly Lipschitz for $l > \lambda(u_1)$ and $s \in [t, M]$, and (iii) as previously proved, the map $u \mapsto L(s, u; r)$ is uniformly Lipschitz across $s \in [t, M]$ and $u \geq 0$, this implies that the map $u \mapsto x_{r,u,t}(s) - s$ is Lipschitz across $s \in K$ and $u \geq u_1$. Next note that the map from $d \mapsto \frac{1}{d}$ is Lipschitz for $d > \tilde{c}$. With the fact that $x_{r,u,t}(s) - s \geq \tilde{c}$ for $s \in K$, and the previous Lipschitz conclusions, this implies that the map

$$u \mapsto \exp\left(-\int_{\alpha_j}^{s} \frac{dv}{x_{r,u,t}(v) - v}\right)$$

is uniformly Lipschitz across $s \in K$ and $u \geq u_1$. But, combining this with the fact that $L_u(s, u; r)$ is uniformly bounded by w^J, this means that there exists a $\gamma > 0$ such that for any $j = 0, ..., J$, any $s \in K_j$, and any $\overline{u} > \underline{u} \geq u_1$,

$$\begin{aligned}
|L_u(s, \overline{u}; r) - L_u(s, \underline{u}; r)| =& |L_u(\alpha_j, \overline{u}; r) \exp\left(-\int_{\alpha_j}^{s} \frac{dv}{x_{r,\overline{u}}(v) - v}\right) - L_u(\alpha_j, \underline{u}; r) \exp\left(-\int_{\alpha_j}^{s} \frac{dv}{x_{r,\underline{u}}(v) - v}\right)| \\
=& |\left(L_u(\alpha_j, \overline{u}; r) - L_u(\alpha_j, \underline{u}; r)\right)| \exp\left(-\int_{\alpha_j}^{s} \frac{dv}{x_{r,\overline{u}}(v) - v}\right) \\
&+ |L_u(\alpha_j, \underline{u}; r)| \left| \exp\left(-\int_{\alpha_j}^{s} \frac{dv}{x_{r,\overline{u}}(v) - v}\right) - \exp\left(-\int_{\alpha_j}^{s} \frac{dv}{x_{r,\underline{u}}(v) - v}\right) \right| \\
\leq& |L_u(\alpha_j, \overline{u}; r) - L_u(\alpha_j, \underline{u}; r)| + \gamma(\overline{u} - \underline{u}),
\end{aligned}$$

where the last inequality uses that $\exp(-v) \leq 1$ for any $v > 0$. Now consider $s \in [a_j, b_j]$ so that

$$L_u(s, u; r) = L_u(a_j, u; r)\frac{x_u^j - s}{x_u^j - a_j},$$

where $x_u^j = \sqrt{L(a_j, u; r) + \rho(R(a_j, b_j) - a_j)} + a_j$ This means that again since $L(a_j, u; r) > \lambda(u_1)$, and $x_u^j - a_j > \tilde{c}$, the mapping $u \mapsto \frac{x_u^j - s}{x_u^j - a_j}$ is uniformly Lipschitz across $s \in \cup_{j=1}^J I_j$ and $u \geq u_1$. Thus, But, combining this with the fact that $L_u(s, u; r)$ is uniformly bounded by w^J, there exists an $\varphi > 0$ such that for any $j = 1, ..., J$ and $\overline{u} > \underline{u} \geq u_1$, such that

$$\begin{aligned}
|L_u(s, \overline{u}; r) - L_u(s, \underline{u}; r)| =& \left| L_u(a_j, \overline{u}; r)\frac{x_{\overline{u}}^j - s}{x_{\overline{u}}^j - a_j} - L_u(a_j, \underline{u}; r)\frac{x_{\underline{u}}^j - s}{x_{\underline{u}}^j - a_j} \right| \\
=& |L_u(a_j, \overline{u}; r) - L_u(a_j, \underline{u}; r)| \left| \frac{x_{\overline{u}}^j - s}{x_{\overline{u}}^j - a_j} \right| + |L_u(a_j, \overline{u}; r)| \left| \frac{x_{\overline{u}}^j - s}{x_{\overline{u}}^j - a_j} - \frac{x_{\underline{u}}^j - s}{x_{\underline{u}}^j - a_j} \right| \\
\leq& |L_u(a_j, \overline{u}; r) - L_u(a_j, \underline{u}; r)| \max\{M/\tilde{c}, 1\} + \varphi(\overline{u} - \underline{u}).
\end{aligned}$$

The bound $M/\tilde{c} > \left| \frac{x_{\overline{u}}^j - s}{x_{\overline{u}}^j - a_j} \right|$ comes from the fact that $x_{\overline{u}}^j - s > -M$ and $x_{\overline{u}}^j - a_j > \tilde{c}$ as previously proven.

Using (i) these two bounds on $|L_u(s, \overline{u}; r) - L_u(s, \underline{u}; r)|$, (ii) the fact that $L(t, u; r) = u \; \forall u$ so $|L_u(t, \overline{u}; r) - L_u(t, \underline{u}; r)| = 0$, and (iii) iterating through the at most J pooling intervals and at most $J+1$ connected components of K, provides for a constant $\zeta > 0$ such that gives

$$|L_u(s, \overline{u}; r) - L_u(s, \underline{u}; r)| \leq \zeta(\overline{u} - \underline{u}).$$

So the family $\{L_u(s, u; r)\}_{r \in \mathcal{R}}$ is Uniformly equicontinuous for $u \geq u_1$ and $s \in [t, M]$

For each reputation function r, define

$$G(r, u) := \frac{1}{F([t, M])} \int_t^M L(s, u; r) f(s)\, ds.$$

Then the family

$$\{G_u(r, \cdot)\}_r$$

is equicontinuous on $u \geq u_1$ because the family $L_u(s, \cdot; r)_{r,s}$ is equicontinuous on $u \geq u_1$. This means that the family $G(r, \cdot)\}_r$ is equidifferentiable on $u \geq u_1$. Theorem 3 in Milgrom and Segal (2002) then provides the result.

Q.E.D.

Lemma 14. *Consider an optimal allocation x^* with $\underline{t} \equiv \underline{\tau}(t) < \overline{\tau}(t) \equiv \overline{t}$ for some t. Let $L^A(\underline{t}|x^*) \equiv u_0$, $L^A(\overline{t}|x^*) \equiv u_1$, $x^*(\underline{t}) \equiv \tilde{x}$, $R(\underline{t},\overline{t}) \equiv \tilde{R}$, and $F([\underline{t},\overline{t}]) \equiv \tilde{F}$. It holds that,*

$$\frac{2\tilde{F}(\tilde{x}-\tilde{R})}{\tilde{x}-\underline{t}+\sqrt{u_0}} - (\tilde{R}-\underline{t})f(\underline{t}) + \frac{F([\overline{t},M])}{\tilde{F}} DV(t,u_1)\left(\frac{2\tilde{F}(\tilde{x}-\overline{t})}{\tilde{x}-\underline{t}+\sqrt{u_0}} - (\overline{t}-\underline{t})f(\underline{t})\right) \geq 0,$$

where

$$DV(t,u) \equiv \begin{cases} V_{u^-}(t,u) & \textit{if } \frac{2\tilde{F}(\tilde{x}-\overline{t})}{\tilde{x}-\underline{t}+\sqrt{u_0}} - (\overline{t}-\underline{t})f(\underline{t}) < 0 \\ V_{u^+}(t,u) & \textit{if } \frac{2\tilde{F}(\tilde{x}-\overline{t})}{\tilde{x}-\underline{t}+\sqrt{u_0}} - (\overline{t}-\underline{t})f(\underline{t}) \geq 0 \end{cases}.$$

Proof. Consider the continuing allocation $x^*_{|[\underline{t},M]}$ at $(\underline{t},u_0)$. I will show that the inequality above results from this continuing allocation being optimal at (t,u_0). Consider an alternative continuing allocation at $(\underline{t},u_0)$ parameterized by m that (i) separates between $\underline{t}$ and m, (ii) pools between m and $\overline{t}$, and (iii) chooses an optimal continuing allocation above $\overline{t}$. Let $a^m \equiv \sqrt{D_{u_0}(m-\underline{t})+\rho(R(m,\overline{t})-m)}+m$ and $u^m \equiv (a^m-\overline{t})^2+\rho(\overline{t}-R(m,\overline{t}))$. Finally, let $x^*_{t,u}$ be an optimal continuing allocation at (t,u). Specifically the alternative continuing allocation is defined as

$$y^m \equiv \begin{cases} d_{u_0}(t'-\underline{t})+\underline{t} & \underline{t} \leq t' < m \\ a^m & m \leq t' < \overline{t} \\ x^*_{\overline{t},u^m}(t') & t' \geq \overline{t} \end{cases}.$$

Because x^* is optimal, it must be that $\frac{d\,L^P(y^m)}{dm}\Big|_{m=\underline{t}} \geq 0$. This gives the inequality in the display. *Q.E.D.*

B. Proofs from Section 2

B.1. Proof of Lemma 1

Proof. " $\implies$ " $x(t)$ is increasing because $\forall x : T \to A$, $L^A(a,t|x)$ is strictly submodular in (a,t), so any minimizing selection from $\min_{a\in x(T)} L^A(a,t|x)$ is increasing.

Since $x(t)$ is increasing it has an at most countable set of discontinuities by Froda's theorem, i.e. J_x is countable.

Now suppose J_x is dense on some interval $[\underline{t},\overline{t}]$. First, consider that $x(t_1) = x(t_2)$ for $\underline{t} \leq t_1 < t_2 \leq \overline{t}$. Then since x is increasing $x(t)$ is constant on $[t_1,t_2]$, i.e. $J_x \cap (t_1,t_2) = \emptyset$ contradicting the hypothesis that J_x is dense on this interval. Thus, x is injective on $[\underline{t},\overline{t}]$, and $r^*_x(t) = t\ \forall t \in [\underline{t},\overline{t}]$. Note that $\forall x \in IC(T)$, $L^A(a,t|x)$ is continuous in t and so incentive compatibility implies that $L^A(t|x)$ is also continuous in t . Therefore, since the reputation is continuous on $[\underline{t}, \bar{t}]$, continuity of $L^A(t, |x)$ requires that the action $x(t)$ also be continuous on $(\underline{t}, \bar{t})$. This means $(\underline{t}, \bar{t}) \cap J_x = \emptyset$, which is a contradiction. This continuity of $L^A(t|x)$ also implies the third condition in the lemma.

This means J_x is a countable nowhere dense set. Take arbitrary $\underline{t}, \bar{t} \in J_x$ with $(\underline{t}, \bar{t}) \cap J_x = \emptyset$. First suppose that x is not strictly increasing on $(\underline{t}, \bar{t})$ Then since x is increasing there exists $\underline{t} \leq t_1 < t_2 \leq \bar{t}$ such that $x(t') \equiv \tilde{x}$ is constant $\forall t' \in (t_1, t_2)$, where $t_1 \equiv \inf\{t \in T : \ x(t) = \tilde{x}\}$ and $t_2 \equiv \sup\{t \in T : \ x(t) = \tilde{x}\}$. Suppose $t_1 > \underline{t}$. Then $\forall \varepsilon > 0$, $r_x(\tilde{x}) - r_x(x(t_1 - \varepsilon)) > R(t_1, t_2) - t_1$. Since $L^A(t|x)$ is continuous in t and the reputation discontinuously jumps at t_1, the action must also jump to preserve IC. But this contradicts the fact that $(\underline{t}, \bar{t}) \cap J_x = \emptyset$, so $t_1 = \underline{t}$. A symmetric argument shows that $t_2 = \bar{t}$, and so x is constant on $(\underline{t}, \bar{t})$. This means that x is either constant or strictly increasing on $(\underline{t}, \bar{t})$.

Suppose x is strictly increasing on $(\underline{t}, \bar{t})$. Then x is injective on this interval and $r_x(x(t)) = t \ \forall t \in (\underline{t}, \bar{t})$. And so

$$L^A(x(t'), t|x) = (x(t') - t)^2 + \rho(t - t').$$

For $\varepsilon > 0$, incentive compatibility implies,

$$\begin{aligned} & L^A(x(t+\varepsilon), t|x) - L^A(x(t), t|x) \geq 0 \\ \Longrightarrow \ & x(t+\varepsilon)^2 - x(t)^2 - 2t(x(t+\varepsilon) - x(t)) \geq \rho\varepsilon \\ \Longrightarrow \ & \frac{x(t+\varepsilon) - x(t)}{\varepsilon} \geq \frac{\rho}{x(t) + x(t+\varepsilon) - 2t}. \end{aligned}$$

An analogous argument for $\varepsilon < 0$ applies, so it holds that $\forall \varepsilon > 0$,

$$\frac{\rho}{x(t) + x(t-\varepsilon) - 2t} \geq \frac{x(t+\varepsilon) - x(t)}{\varepsilon} \geq \frac{\rho}{x(t) + x(t+\varepsilon) - 2t}.$$

Since $x(t+\varepsilon)$ is continuous in ε for $|\varepsilon|$ small. $x'(t) = \frac{\rho}{2(x(t)-t)}$.

" $\Longleftarrow$ " Since $L^A(a, t|x)$ is submodular, local incentive constraints are sufficient for global incentive constraints.[36] First consider $t \notin J_x$. Either x is constant around t or solves (2b) which is defined by preserving local incentives. For $t \in J_x$ condition 3 is equivalent to local incentive compatibility. *Q.E.D.*

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[36] This fact is detailed in Carroll (2012).

B.2. Proof of Lemma 2

Proof. The solution to (2b) exists and is given by

$$d_u(t) \equiv t + \rho/2\left(1 + W_0\left(-e^{-\frac{\rho-2(\sqrt{u}-t)}{\rho}}\frac{\rho-2\sqrt{u}}{\rho}\right)\right),$$

Where $W_0(z)$ is the Lambert W-function, i.e. the principal solution to $z = W_0(z)e^{W_0(z)}$. The properties of the lemma come directly from examining (2b) as follows.

Since $D_u(t) = (d_u(t) - t)^2$ and $d_u'(t) = \rho/[2(d_u(t) - t)]$, we have

$$D_u'(t) = 2(d_u(t) - t)(d_u'(t) - 1) = \rho - 2\sqrt{D_u(t)}, \qquad D_u(0) = u.$$

This admits a solution that is continuously differentiable because $\varphi(y) \equiv \rho - 2\sqrt{y}$ is continuous for $y \in \mathbb{R}_+$. Hence $D_u'(t) \geq 0$ if and only if $D_u(t) \leq \rho^2/4$, with equality only when $D_u(t) = \rho^2/4$. Because $\varphi(y)$ is locally Lipschitz around $y = \rho^2/4$, one obtains uniqueness of the solution to the ODE given that $D_u(t) = \rho^2/4$ at some t. Since the constant loss $D_u(t) = \rho^2/4$ is such a solution, $D_u(t)$ cannot cross $\rho^2/4$ at any t. Therefore, if $u \leq \rho^2/4$, then $D_u(t) \leq \rho^2/4$ for all t, so D_u is increasing; if $u \geq \rho^2/4$, then $D_u(t) \geq \rho^2/4$ for all t, so D_u is decreasing.

Differentiating once more,

$$D_u''(t) = -\frac{D_u'(t)}{\sqrt{D_u(t)}},$$

so D_u is concave in the first case and convex in the second.

Since D_u is monotone and bounded on the appropriate side by $\rho^2/4$, it has a limit L. If $L < \rho^2/4$, then eventually $D_u'(t) \geq c > 0$ for some $c > 0$, a contradiction. If $L > \rho^2/4$, then eventually $D_u'(t) \leq -c < 0$ for some $c > 0$, also a contradiction. Thus

$$\lim_{t\to\infty} D_u(t) = \rho^2/4.$$

Finally, if $u_1 < u_2$, then D_{u_1} and D_{u_2} solve the same autonomous ODE with ordered initial conditions. By uniqueness, their graphs cannot intersect, so

$$D_{u_1}(t) < D_{u_2}(t) \qquad \forall t \geq 0.$$

Hence $D_u(t)$ is strictly increasing in u.

Q.E.D.

B.3. Proof of Lemma 3

Proof. By Lemma 12. the set of agent loss functions is compact and since $L^P(x) = \int_0^M L^A(t'|x)f(t')dt'$, the principal's loss is continuous in the agent's loss and therefore a minimum exists.

Q.E.D.

C. Proofs from Section 3

C.1. Proof of Lemma 4

Proof. By Lemma 12 the set of agent loss functions from continuing allocations is compact. Since, for $x \in C(t,u)$, $L^P(x) = \int_t^M L^A(t'|x)f(t')dt'$ is continuous in the agent's loss *Q.E.D.*

C.2. Proof of Proposition 1

Proof. First I establish two claims.

Claim 5. *For any $s > t$, and two allocations x_1 and x_2 with the same reputation function r, if $L^A(t'|x_1) - L^A(t'|x_2) \geq 0 \; \forall t' \in [t,s]$ then $L^A(t'|x_1) - L^A(t'|x_2)$ is decreasing for all $t' \in [t,s]$.*

Proof of Claim: Note that by Lemma 8 for $s \geq t_2 > t_1 \geq t$,

$$\begin{aligned} & L^A(t_2|x_1) - L^A(t_2|x_2) - (L^A(t_1|x_1) - L^A(t_1|x_2)) \\ =& \int_{t_1}^{t_2} 2(x_2(\tilde{t}) - x_1(\tilde{t}))d\tilde{t} \\ =& \int_{t_1}^{t_2} 2\left(\sqrt{L^A(\underline{\tau}(\tilde{t})|x_2) + \rho(r(\tilde{t}) - \underline{\tau}(\tilde{t}))} - \sqrt{L^A(\underline{\tau}(\tilde{t})|x_1) + \rho(r(\tilde{t}) - \underline{\tau}(\tilde{t}))}\right) d\tilde{t} \leq 0, \end{aligned}$$

where the last inequality follows from the hypothesis that $L^A(t'|x_1) - L^A(t'|x_2) \geq 0 \; \forall t' \in [t,s]$. This proves the claim.

Recall from Lemma 11 that $x_{r,u,t}$ is the continuing allocation in $C(t,u)$ with reputation function r. Let $\beta = \inf_{t_2 > t_1} \frac{R(t_2,t_1) - t_1}{t_2 - t_1}$ where $1 > \beta > 0$ because the density $f(t) \in [\underline{k}, \overline{k}]$, where $\underline{k} > 0$.

Claim 6. *There exists $\tilde{\delta}$ such that for any reputation function r on T, $\forall t, s \in T$, with $t \in \tau(T)$, $\tilde{\delta} > s - t > 0$, and $u > \varepsilon > 0$, it holds that $L^A(s|x_{r,u,t}) - L^A(s|x_{r,u-\varepsilon,t}) > \varepsilon/2$.*

Proof of Claim: I prove the claim for an arbitrary t and r and note that our construction of $\tilde{\delta}$ does not depend either. To ease notation let $x_u \equiv x_{r,u,t}$.

Now towards a contradiction suppose there exists a sequence of $\varepsilon_n > 0$, $u_n > \varepsilon_n$ and $s_n \to t^+$ such that $L^A(s_n|x_{u_n}) - L^A(s_n|x_{u_n-\varepsilon_n}) \leq \varepsilon_n/2$. Without loss, take $s_n - t < \overline{\delta}$ $\forall n$. In addition, since $L^A(t|x_{u_n}) - L^A(t|x_{u_n-\varepsilon_n}) = \varepsilon_n > 0$, and the agent's equilibrium loss is continuous in his type, it is without loss to take s_n to be the first violator, i.e., so that $L^A(s_n|x_{u_n}) - L^A(s_n|x_{u_n-\varepsilon_n}) = \varepsilon_n/2$ and $\forall t' \in [t, s_n]$ $L^A(t'|x_{u_n}) - L^A(t'|x_{u_n-\varepsilon_n}) \geq \varepsilon/2$. By Claim 5, $L^A(t'|x_{u_n}) - L^A(t'|x_{u_n-\varepsilon_n})$ is decreasing in t' for $t' \in [t, s_n]$ and in particular $L^A(t'|x_{u_n}) - L^A(t'|x_{u_n-\varepsilon_n}) < \varepsilon_n$. Using Lemma 8,

$$\begin{aligned}
&L^A(s_n|x_{u_n}) - L^A(s_n|x_{u_n-\varepsilon_n}) \\
=&\varepsilon_n + \int_t^{s_n} 2(x_{u_n-\varepsilon_n}(\tilde{t}) - x_{u_n}(\tilde{t}))d\tilde{t} \\
=&\varepsilon_n + \int_t^{s} 2\left(\sqrt{L^A(\underline{\tau}(\tilde{t})|x_{u_n-\varepsilon_n}) + \rho(r(\tilde{t}) - \underline{\tau}(\tilde{t}))} - \sqrt{L^A(\underline{\tau}(\tilde{t})|x_{u_n}) + \rho(r(\tilde{t}) - \underline{\tau}(\tilde{t}))}\right) d\tilde{t} \\
>&\varepsilon_n - \int_t^{s} \frac{\varepsilon_n}{\sqrt{L^A(\underline{\tau}(\tilde{t})|x_{u_n-\varepsilon_n}) + \rho\beta(\tilde{t} - \underline{\tau}(\tilde{t}))}} d\tilde{t} \\
>&\varepsilon_n - \int_t^{s} \frac{\varepsilon_n}{\sqrt{\left(\sqrt{u_n - \varepsilon_n + \rho(\underline{\tau}(\tilde{t}) - t)} - (\underline{\tau}(\tilde{t}) - t)\right)^2 + \rho\beta(\tilde{t} - \underline{\tau}(\tilde{t}))}} d\tilde{t}
\end{aligned}$$

The first inequality follows from the concavity of the square root function, that $r^*(\tilde{t}) - \underline{\tau}(\tilde{t}) > \beta(\tilde{t} - \underline{\tau}(\tilde{t}))$ by the definition of β and that $r^*(\tilde{t}) \geq R(\underline{\tau}(\tilde{t}), \tilde{t})$, and that $L^A(t'|x^*) - L^A(t'|x_{u_n-\varepsilon_n}) < \varepsilon_n/2$ by the argument above. The last inequality uses Lemma 10 and the fact that $s_n - t \leq \overline{\delta}$. Now note that there exists $\hat{s} > t$, such that for all $\varepsilon > 0$ $u > \varepsilon$, and for all $\tilde{t} \in [t, \hat{s}]$,

$$\begin{aligned}
&\left(\sqrt{u - \varepsilon + \rho(\underline{\tau}(\tilde{t}) - t)} - (\underline{\tau}(\tilde{t}) - t)\right)^2 + \rho\beta(\tilde{t} - \underline{\tau}(\tilde{t})) \\
>&\left(\sqrt{\rho(\underline{\tau}(\tilde{t}) - t)} - (\underline{\tau}(\tilde{t}) - t)\right)^2 + \rho\beta(\tilde{t} - \underline{\tau}(\tilde{t})) \\
=&\rho(1-\beta)(\underline{\tau}(\tilde{t}) - t) - 2\left(\sqrt{\rho(\underline{\tau}(\tilde{t}) - t)}\right) + (\underline{\tau}(\tilde{t}) - t)^2 + \rho\beta(\tilde{t} - t) \\
=&(\underline{\tau}(\tilde{t}) - t)\left(\rho(1-\beta) - 2\left(\sqrt{\rho(\underline{\tau}(\tilde{t}) - t)}\right) + (\underline{\tau}(\tilde{t}) - t)\right) + \rho\beta(\tilde{t} - t) \\
>&\rho\beta(\tilde{t} - t),
\end{aligned}$$

where the first inequality comes from taking $\hat{s} - t < \overline{\delta}$ so that $\sqrt{\rho(\underline{\tau}(\tilde{t}) - t)} > \underline{\tau}(\tilde{t}) - t$, and the second inequality comes from noting that when $\underline{\tau}(\tilde{t}) - t$ becomes small as $\tilde{t} \to t$, the term in large parentheses becomes positive because $\beta < 1$. Thus by passing to a subsequence such that $s_n < \hat{s}$, using this expression in the display above gives that

$$\begin{aligned}&L^A(\tilde{s}|x_{u_n}) - L^A(\tilde{s}|x_{u_n-\varepsilon_n})\\ >&\varepsilon_n - \int_t^{s_n} \frac{\varepsilon_n}{\sqrt{\rho\beta(\tilde{t}-t)}}d\tilde{t}\\ >&\varepsilon_n(1-2\sqrt{\rho\beta(s_n-t)}).\end{aligned}$$

But the last expression converges to ε_n as $s_n \to t$ contradicting the fact that $L^A(s_n|x_{u_n}) - L^A(s_n|x_{u_n-\varepsilon_n}) = \varepsilon_n/2$ and proving the claim.

Now let x^* be optimal in $C(t,u)$. Define $\tilde{L}(t') = (x^*(t') - t')^2 + \rho(t' - R(t', \overline{\tau}(t')))$ as the agent's loss at t' using the same action as x^* but with the reputation from only pooling with higher types in his pooled set. Construct three recursive sequences $\{u_i\}, \{t_i\}, \{s_i\}$ as follows. Let $t_0 = s_0 \equiv t$ and $u_0 \equiv u - \varepsilon$. Let s_{i+1} be the first type $t' \geq \overline{\tau}(t_i^-)$ such that $L^A(t'|x_{r^*,u_i,\overline{\tau}(t_i^-)}) = L^A(t'|x^*)$. If no such s_{i+1} exists, then set $s_{i+1} = t_{i+1} \equiv M$ and conclude the sequence at $i = n$. If it does exist, for $t' \in (\underline{\tau}(s_{i+1}^-), s_{i+1})$ let

$$\begin{aligned}a_{i+1}(t') \equiv&\sqrt{L^A(\underline{\tau}(s_{i+1}^-)|x_{r^*,u_i,\overline{\tau}(t_i^-)}) + \rho(R(\underline{\tau}(s_{i+1}^-), t') - \underline{\tau}(s_{i+1}^-))} + \underline{\tau}(s_{i+t}^-), \text{ and}\\ \hat{L}(t') \equiv&(a_{i+1}(t') - t')^2 + \rho(t' - R(\underline{\tau}(s_{i+1}^-), t'))\end{aligned}$$

be the continuing action for this interval given a lower truncated reputation at t' and the associated loss at t' from choosing this action and pooling with this truncated interval. Let t_{i+1} be the first type $t' \in (\underline{\tau}(s_{i+1}^-), s_{i+1})$ to solve

$$\tilde{L}(t') = \hat{L}(t'). \tag{14}$$

Now set $u_{i+1} \equiv (x^*(\overline{\tau}(t_{i+1})^-) - \overline{\tau}(t_{i+1}^-))^2 + \rho(\overline{\tau}(t_{i+1}) - R(t_{i+1}, \overline{\tau}(t_{i+1})))$ as the loss at the next threshold after t_{i+1} after t_{i+1} under r^* given x^* and the reputation from pooling with the truncated interval between t_{i+1} and this next threshold.

I will show by induction that (i) $u_i < L^A(\overline{\tau}(t_i^-)|x^*)$, (ii) t_{i+1} is well defined for every $n \geq i \geq 0$, and (iii) that $\underline{\tau}(s_i^-) + \tilde{\delta} < s_i$ for $n \geq i \geq 1$, where $\tilde{\delta} > 0$ is from Claim 6.

The definition of u_0, t_0, and s_0 establishes the base step. Because the agent's loss is continuous in his type, the induction hypothesis that $L^A(\overline{\tau}(t_i^-)|x_{r^*,u_i,\overline{\tau}(t_i^-)}) = u_i < L^A(\overline{\tau}(t_i^-)|x^*)$, and the definition of s_{i+1}, it holds that

$$L^A(t'|x_{r^*,u_i,\overline{\tau}(t_i^-)}) < L^A(t'|x^*) \; \forall \, t' \in [\overline{\tau}(t_i^-), s_{i+1}]. \tag{15}$$

Now suppose towards a contradiction to the third point that there exists a threshold $\underline{s} = \underline{\tau}(\underline{s}) < s_{i+1}$ such that $\underline{s} + \tilde{\delta} \geq s_{i+1}$. Define $\varepsilon \equiv L^A(\underline{s}|x^*) - L^A(\underline{s}|x_{r^*,u_i,\overline{\tau}(t_i^-)})$ which is positive because $\underline{s} < s_{i+1}$. Now note that,

$$\begin{aligned}
0 =& L^A(s_{i+1}|x^*) - L^A(s_{i+1}|x_{r^*,u_i,\overline{\tau}(t_i^-)}) \\
=& L^A(s_{i+1}|x_{r^*,L^A(\underline{s}|x^*),\underline{s}}) - L^A(s_{i+1}|x_{r^*,L^A(\underline{s}|x^*)-\varepsilon,\underline{s},}) \\
>& \varepsilon/2,
\end{aligned}$$

which is a contradiction. The first equality just tracks the continuing allocations to start from the threshold $\underline{s}$ and its associated loss. The inequality comes from Claim 6 and the hypothesis that $\underline{s} + \tilde{\delta} \geq s_{i+1}$.

Now I will show the second point that t_{i+1} is well defined. Notice that, at $t' = \underline{\tau}(s_{i+1}^-)$, the RHS of (14) evaluates to $L^A(\underline{\tau}(s_{i+1}^-)|x_{r^*,u_i,\overline{\tau}(t_i^-)})$ and the LHS evaluates to $L^A(\underline{\tau}(s_{i+1}^-)|x^*)$. So, because $\underline{\tau}(s_{i+1}^-) < s_{i+1}$, (15) gives that the LHS>RHS of (14) at the left boundary of the interval. To show the opposite inequality at s_{i+1} note that

$$\hat{L}(s_{i+1}) \geq L^A(\underline{\tau}(s_{i+1}^-)|x_{r^*,u_i,\overline{\tau}(t_i^-)}) = L^A(s_{i+1}|x^*) > \tilde{L}(s_{i+1}) \tag{16}$$

The first inequality holds because of the following reason. As t' is increased $a_{i+1}(t')$ is increased to exactly compensate the reputational increase from pooling with higher types for type $\underline{\tau}(s_{i+1}^-)$. Because of the sub-modularity of the agent's material loss, this exchange is positive for any higher types like s_{i+1}, and so

$$(a_{i+1}(s_{i+1})-s_{i+1})^2+\rho(s_{i+1}-R(\underline{\tau}(s_{i+1}^-), s_{i+1})) \geq (a_{i+1}(\overline{\tau}(s_{i+1}^-))-s_{i+1})^2+\rho(s_{i+1}-R(\underline{\tau}(s_{i+1}^-), \overline{\tau}(s_{i+1}^-))),$$

which is the expanded version of the first inequality in (16). The equality in (16) comes from the definition of s_{i+1}. The last inequality in (16) is because $\tilde{L}(s_{i+1})$ is the loss for s_{i+1} under the same action assignment as x^*, but with a right censored higher reputation at s_{i+1}. The strict inequality comes from the fact, as argued above, that $s_{i+1} > \underline{\tau}(s_{i+1}^-)$. Thus because both sides of (14) are continuous in t' and the LHS>RHS at the left boundary while the LHS<RHS at the right boundary, the mean value theorem guarantees t_{i+1} is well defined.

Finally I show point (i). Note that since $t_{i+1} \in (\underline{\tau}(s_{i+1}^-), s_{i+1})$, it holds that $t_{i+1} > \underline{\tau}(t_{i+1})$,

$$\begin{aligned}
L^A(\overline{\tau}(t_{i+1}^-)|x^*) =& (x^*(\overline{\tau}(t_{i+1})^-) - \overline{\tau}(t_{i+1}^-))^2 + \rho(\overline{\tau}(t_{i+1}) - R(\underline{\tau}(t_{i+1}), \overline{\tau}(t_{i+1}))) \\
>& (x^*(\overline{\tau}(t_{i+1})^-) - \overline{\tau}(t_{i+1}^-))^2 + \rho(\overline{\tau}(t_{i+1}) - R(t_{i+1}, \overline{\tau}(t_{i+1}))) \\
=& u_{i+1}
\end{aligned}$$

with the last equality just using the definition of u_{i+1}.

Thus it only remains to show that the sequence concludes in finite steps. But

$$t_i < \overline{\tau}(t_i) \leq \underline{\tau}(s_{i+1}^-) < \underline{\tau}(s_{i+1}^-) + \tilde{\delta} < s_{i+1} \leq \overline{\tau}(t_{i+1}^-) < t_{i+2}.$$

So the sequence grows by at least $\tilde{\delta}$ every two iterations implying that we exhaust the type space in finite steps.

Now define a continuing allocation y at $(t, u - \varepsilon)$ as

$$y(t') = \begin{cases} x_{r^*,u_i,\overline{\tau}(t_i^-)}(t') & t' \in [\overline{\tau}(t_i^-), \underline{\tau}(t_{i+1}) \\ a_{i+1}(t_{i+1}) & t' \in [\underline{\tau}(t_{i+1}), t_{i+1}) \\ x^*(t') & t' \in [t_{i+1}, \overline{\tau}(t_{i+1}^-)) \end{cases} .$$

This is a well defined continuing allocation by construction. Note that $\forall t' \in [t, M]$ $L^A(t'|x^*) > L^A(t'|y)$. To see this consider the three cases in the definition above. First, recall that, by the definition of s_{i+1}, $L^A(t'|x_{r^*,u_i,\overline{\tau}(t_i^-)}) < L^A(t'|x^*)$ for $t' \in [\overline{\tau}(t_i^-), s_{i+1})$. By the definition of y, $L^A(t'|y) = L^A(t'|x_{r^*,u_i,\overline{\tau}(t_i^-)})$ for $t' \in [\overline{\tau}(t_i^-), \underline{\tau}(t_{i+1})$. Since $t_{i+1} \in (\underline{\tau}(s_{i+1}^-), s_{i+1})$ by construction, $\underline{\tau}(t_{i+1}) < s_{i+1}$ establishing the comparison for this region.

Second, for $t' \in [\underline{\tau}(t_{i+1}), t_{i+1})$, note that by Lemma 8

$$\begin{aligned} & L^A(t'|x^*) - L^A(t'|y) \\ =& L^A(t_{i+1}|x^*) - L^A(t_{i+1}|y) + \int_{t'}^{t_{i+1}} 2(x^*(t_{i+1}) - a_{i+1}(t_{i+1}))d\tilde{t} \\ \geq& L^A(t_{i+1}|x^*) - L^A(t_{i+1}|y) \\ =& R(t_{i+1}, \overline{\tau}(t_{i+1}) - R(\underline{\tau}(t_{i+1}), \overline{\tau}(t_{i+1})) > 0. \end{aligned}$$

The first inequality comes from the fact the fact that $a_{i+1}(t_{i+1}) < x^*(t_{i+1}^-)$ which follows from the definition and the fact that $L^A(\underline{\tau}(s_{i+1}^-)|x_{r^*,u_i,\overline{\tau}(t_i^-)}) < L^A(\underline{\tau}(s_{i+1}^-)|x^*)$. The second equality comes from the construction of t_{i+1}, and the final inequality is because $t_{i+1} > \underline{\tau}(s_{i+1}^-) = \underline{\tau}(t_{i+1}^-)$.

Third, and similarly for $t' \in [t_{i+1}, \overline{\tau}(t_{i+1}^{-}))$, note that by Lemma 8

$$\begin{aligned}
&L^A(t'|x) - L^A(t'|y) \\
=&L^A(t_{i+1}|x^*) - L^A(t_{i+1}|y) - \int_{t_{i+1}}^{t'} 2(x^*(t_{i+1}) - y(t_{i+1}))d\tilde{t} \\
=&L^A(t_{i+1}|x^*) - L^A(t_{i+1}|y) \\
=&R(t_{i+1}, \overline{\tau}(t_{i+1}) - R(\underline{\tau}(t_{i+1}), \overline{\tau}(t_{i+1}) > 0.
\end{aligned}$$

The second equality comes from the fact that $y(t') = x^*(t') = x^*(t_{i+1})$ in this region. The other relations follow as above. Thus, $\forall t' \in [t, M]$ $L^A(t'|x^*) > L^A(t'|y)$. Next I establish the following useful claim.

Claim 7. *For $i \geq 1$ and $t_i < M$, let $L^A(\underline{\tau}(t_i)|x^*) - L^A(\underline{\tau}(t_i)|y) \equiv \varepsilon > 0$. Then, $\forall t' \in [\underline{\tau}(t_i), \overline{\tau}(t_i)]$,*

$$L^A(t'|x^*) - L^A(t'|y) \geq \frac{\rho\underline{\beta}\varepsilon}{2B + \rho\overline{\beta}}.$$

Proof of Claim: From the construction of y, $y(t') = x^*(t')$ for $t' \in [t_i, \overline{\tau}(t_i))$ and as described above $y(t') = a_i(t_{i+1}) < x^*(t') = x^*(t_{i+1})$ for $t' \in [\underline{\tau}(t_i), t_i)$. Thus by Lemma 8 for $t' \in [\underline{\tau}(t_i), \overline{\tau}(t_i)]$

$$\begin{aligned}
&L^A(t'|x) - L^A(t'|y) \\
=&L^A(t_i|x^*) - L^A(t_i|y) - \int_{t_i}^{t'} 2(x^*(\tilde{t}) - y(\tilde{t}))d\tilde{t} \\
\geq&L^A(t_i|x^*) - L^A(t_i|y).
\end{aligned}$$

Now recall the definition of t_i as the first $t' \in [\underline{\tau}(t_i), \overline{\tau}(t_i)]$ such that (14) is satisfied. Let $\tilde{u} \equiv L^A(\underline{\tau}(t_i)|x^*)$ Expanding (14), using the definition of $x^*(t_i)$ and $a_i(t_i)$ gives

$$\begin{aligned}
&(a_i(t_i) - t_i)^2 + \rho(t_i - R(\underline{\tau}(t_i), t_i)) = (x^*(t_i) - t_i)^2 + \rho(t_i - R(t_i, \overline{\tau}(t_i^-))) \\
\iff& \varepsilon - 2(t_i - \underline{\tau}(t_i))(x^*(t_i) - a_i(t_i)) = \rho(R(t_i, \overline{\tau}(t_i^-)) - R(\underline{\tau}(t_i), \overline{\tau}(t_i^-))) \\
\implies& \varepsilon - 2B(t_i - \underline{\tau}(t_i)) < \rho\overline{\beta}(t_i - \underline{\tau}(t_i)) \\
\implies& t_i - \underline{\tau}(t_i) > \frac{\varepsilon}{2B + \rho\overline{\beta}}.
\end{aligned}$$

Now, because $x^*(t_i) = y(t_i)$,

$$\begin{aligned}
&L^A(t_i|x^*) - L^A(t_i|y)\\
=&\rho(R(t_i, \overline{\tau}(t_i^-)) - R(\underline{\tau}(t_i), \overline{\tau}(t_i^-)))\\
\geq&\rho\underline{\beta}(t_i - \underline{\tau}(t_i))\\
\geq&\frac{\rho\underline{\beta}\varepsilon}{2B + \rho\overline{\beta}},
\end{aligned}$$

which proves the claim.

I finally proceed to derive the bound in the proposition. Note that $\forall s > t$

$$\begin{aligned}
&\frac{V(t,u) - V(t,u-\varepsilon)}{\varepsilon}\\
\geq&\frac{1}{F[t,M]}\frac{\int_t^M \left((x^*(\tilde{t}) - \tilde{t})^2 - (y(t) - \tilde{t})^2\right) f(\tilde{t})d\tilde{t}}{\varepsilon}\\
=&\frac{1}{F[t,M]}\frac{\int_t^M \left(L^A(\tilde{t}|x^*) - L^A(\tilde{t}|y)\right) f(\tilde{t})d\tilde{t}}{\varepsilon}\\
\geq&\frac{1}{F[t,M]}\frac{\int_t^s \left(L^A(\tilde{t}|x^*) - L^A(\tilde{t}|y)\right) f(\tilde{t})d\tilde{t}}{\varepsilon}.
\end{aligned} \tag{17}$$

The first inequality holds because y is a feasible continuing allocation at $(t, u-\varepsilon)$ while x^* is optimal at (t,u). The equality follows from all allocations having the same average reputation. And the last inequality follows from, as established above, $\forall t' \in [t, M]$ $L^A(t'|x^*) > L^A(t'|y)$. First suppose that $M - t > \tilde{\delta}$, where $\tilde{\delta}$ is from Claim 6. Furthermore suppose first that $t_1 = M$ or $\underline{\tau}(t_1) \geq t + \tilde{\delta}$. Note that by Claim 6 $\forall \tilde{t} \in [t, t+\tilde{\delta}]$

$$L^A(\tilde{t}|x^*) - L^A(\tilde{t}|y) = L^A(\tilde{t}|x^*) - L^A(\tilde{t}|x_{r^*,u-\varepsilon,t}) > \varepsilon/2.$$

Thus taking $s = t + \tilde{\delta}$ in (17) gives

$$\begin{aligned}
&\frac{1}{F[t,M]}\frac{\int_t^{t+\tilde{\delta}} \left(L^A(\tilde{t}|x^*) - L^A(\tilde{t}|y)\right) f(\tilde{t})d\tilde{t}}{\varepsilon}\\
\geq&\frac{1}{\overline{k}M}\frac{\int_t^{t+\tilde{\delta}}(\varepsilon\underline{k}/2)d\tilde{t}}{\varepsilon}\\
=&\frac{\underline{k}\tilde{\delta}}{2\overline{k}M} > 0,
\end{aligned}$$

where the first inequality uses that $f(t) \in [\underline{k}, \overline{k}]$ and the bound on the difference in loss in the previous display. This completes the proof of the proposition in this case.

Next suppose that $t_1 < M$ and $\underline{\tau}(t_1) < t + \tilde{\delta}$. Then, take $s = \overline{\tau}(t_1^-)$ in (17)

$$\begin{aligned}
&\frac{1}{F[t,M]}\frac{\int_t^{\overline{\tau}(t_1^-)}\left(L^A(\tilde{t}|x^*) - L^A(\tilde{t}|y)\right)f(\tilde{t})d\tilde{t}}{\varepsilon}\\
\geq&\frac{1}{F[t,M]}\left(\frac{\int_{\underline{\tau}(t_1)}^{\overline{\tau}(t_1^-)}\left(L^A(\tilde{t}|x^*) - L^A(\tilde{t}|y)\right)f(\tilde{t})d\tilde{t}}{\varepsilon}\right)\\
\geq&\frac{1}{\overline{k}M}\left(\frac{\int_{\underline{\tau}(t_1)}^{\overline{\tau}(t_1^-)}\frac{\underline{k}\rho\underline{\beta}\varepsilon/2}{2B+\rho\overline{\beta}}d\tilde{t}}{\varepsilon}\right)\\
\geq&\frac{\tilde{\delta}\underline{k}\rho\underline{\beta}}{(\overline{k}M)(2B+\rho\overline{\beta})} > 0.
\end{aligned}$$

The first inequality uses that $\forall t' \in [t, M]$ $L^A(t'|x^*) > L^A(t'|y)$. The second inequality uses Claim 6 and the fact that $\underline{\tau}(t_1) < \tilde{\delta} + t$ in this case to give that $L^A(\underline{\tau}(t_1)|x^*) - L^A(\underline{\tau}(t_1)|y) \geq \varepsilon/2$ which allows the use of Claim 7 to bound the integrand (along with the fact that $f(t) \in [\underline{k}, \overline{k}]$). The last inequality uses, as shown in the induction argument point (iii) above, that $\underline{\tau}(s_1) + \tilde{\delta} < s_1 \leq \overline{\tau}(s_1^-)$ and that $\underline{\tau}(t_1) = \underline{\tau}(s_1)$ and $\overline{\tau}(t_1^-) = \overline{\tau}(s_1^-)$ by construction. This proves the proposition in this case.

Now suppose that $t > M - \tilde{\delta}$. Because $t_1 < M$ implies that $t \leq \underline{\tau}(t_1) < t_1 - \tilde{\delta}$, the process must complete in one step. Therefore, by Claim 6 $\forall t' \in [t, M]$ $L^A(t'|x^*) - L^A(t'|y) > \varepsilon/2$. Using this inequality (along with the fact that $f(t) \in [\underline{k}, \overline{k}]$) and $s = M$ in (17) gives

$$\begin{aligned}
&\frac{1}{F[t,M]}\frac{\int_t^{M}\left(L^A(\tilde{t}|x^*) - L^A(\tilde{t}|y)\right)f(\tilde{t})d\tilde{t}}{\varepsilon}\\
\geq&\frac{1}{(M-t)\overline{k}}\frac{\int_t^M(\underline{k}\varepsilon/2)d\tilde{t}}{\varepsilon}\\
=&\frac{\underline{k}}{2\overline{k}} > 0.
\end{aligned}$$

This establishes the proposition in the final case. *Q.E.D.*

C.3. Proof of Theorem 1

Proof. Suppose the theorem does not hold, i.e. there is a sequence of allocations x_n, each one optimal, such that $\lim_{n\to\infty} x_n(0) = 0$. Let $\underline{\tau}_n$ and $\overline{\tau}_n$ be the associated endpoint functions. Note that $\lim_{n\to\infty} \overline{\tau}_n(0) = 0$ as well. This is because $x_n(0) \geq R(0, \overline{\tau}_n(0))$. Otherwise, an alternative allocation that increases $x_n(0)$ to $R(0, \overline{\tau}_n(0))$ improves the principal's loss on the first interval $[0, \overline{\tau}_n(0)]$, and the initial loss of the agent at type $\overline{\tau}_n(0)$. By the alignment principle, this latter change also improves the principal's loss. I begin by breaking the problem into two exhaustive (but not disjoint) cases described below.

Case 1: There exists a pair of subsequences $t_n, s_n \in \overline{\tau}_n(T)$ and $b > 0$ such that $s_n > b$, $(t_n, s_n) \cap \overline{\tau}_n(T) = \emptyset\ \forall n$, and as $n \to \infty$, $t_n \to 0$.

Case 2: There exists a pair of subsequences $t_n, s_n \in \overline{\tau}_n(T)$ such that as $n \to \infty$, $t_n \to 0$, $s_n \to 0$, and $\frac{t_n}{s_n - t_n} \to 0$.

Case 1 captures the case where the sequence of allocations includes an initial vanishing segment $[0, t_n]$ which takes a different lower set of actions than every higher type, and this segment is then followed by a relatively large pool $[t_n, s_n]$. Case 2 captures the case where the allocation is approximately separating for small enough types.

Claim 8. *Either case 1 or case 2 holds.*

Proof of Claim: Take a subsequence $t_n < 1/n^3,\ t_n \in \overline{\tau}_n(T)\ \forall n$ which exists by the fact that $\lim_{n\to\infty} \overline{\tau}_n(0) = 0$. Now suppose that there exists another subsequence s_{n_k} such that $s_{n_k} \in \overline{\tau}_{n_k}(T) \cap (1/n^2, 1/n)$. Then this pair of subsequences satisfies case 2. Suppose that there exists no such subsequence, i.e. $\forall n > N$, $\overline{\tau}_n(T) \cap (1/n^2, 1/n) = \emptyset$. Then take $r_n \equiv \max \overline{\tau}_n(T) \cap [0, 1/n^2]$ and $s_n \equiv \min\ [1/n, M] \cap \overline{\tau}_n(T)$. The subsequences r_n and s_n satisfy case 1 in the case that $s_n \not\to 0$ (by passing to the bounded further subsequence), and case 2 otherwise. This proves the claim.

Suppose case 1 holds. By Lemma 12, the set of IC loss functions $\mathcal{L}$ is compact, and there exists a further subsequence with convergent x_{n_k} and associated convergent s_{n_k} with $s^* \equiv \lim_{n\to\infty} \overline{\tau}_n(b)$ such that x_{n_k} converges to an optimal IC allocation x^* given by

$$x^*(t') = \begin{cases} a & t' < s^* \\ x^*_{s^*,u}(t') & t' \geq s^* \end{cases},$$

where $a \equiv \sqrt{\rho R(0, s^*)}$, $u \equiv (a - s^*)^2 + \rho(s^* - R(0, s^*))$, and $x^*_{s^*,u}$ is an optimal continuing allocation at (s^*, u). This is because both $x^*(t_n^-) \to 0$ and $r_n^*(t_n^-) \to 0$, and by the fact that the assumption that (t_n, s_n) is a pooling interval in case 1 along with incentive compatibility give that $x_n(t) = t_n + \sqrt{(x_n(t_n^-) - t_n)^2 + \rho(R(t_n, s_n) - r_n^*(t_n^-))}\ \forall t \in [t_n, s_n]$.

Now define a new IC allocation as follows parametrized by $r \in [0, s^*]$ as

$$x_r(s) \equiv \begin{cases} d_0(t') & t' < r \\ \sqrt{D_0(r) + \rho(R(r, s^*) - r)} + r & t' \in (r, s^*] \\ x^*_{s^*, u_r}(t') & t' \geq s^* \end{cases}$$

Where $u_r \equiv (\sqrt{D_0(r) + \rho R(r, s^*)} + r - s^*)^2 + \rho(s^* - R(r, s^*))$ and $x^*_{s^*, u_r}(t')$ is an optimal continuing allocation at (s^*, u_r). Notice that for $r = 0$ we recover the value under x^*, i.e., $L^P(x_0) = L^P(x^*)$. Thus, because of the optimality of x^*, it must be the case that, $\left.\frac{d\, L^P(x_r)}{dr}\right|_{r=0} \geq 0$. Evaluating and simplifying this derivative gives,

$$\begin{aligned} &\frac{2}{a}\bigg((a - R(0, s^*))F([0, s^*]) + (a - s^*)V_{u^-}(s^*, u)F([s^*, M])\bigg) \\ \geq &f(0)\left(R(0, s^*) + s^* \frac{F([s^*, M])}{F([0, s^*])} V_{u^-}(s^*, u)\right). \end{aligned} \tag{18}$$

Now define an alternative allocation parameterized by $\tilde{a}$ given by

$$x^{\tilde{a}}(t') = \begin{cases} \tilde{a} & t' < s^* \\ x^*_{s^*, u^{\tilde{a}}}(t') & t' \geq s^* \end{cases},$$

where $u^{\tilde{a}} \equiv (\tilde{a} - s^*)^2 + \rho(s^* - R(0, s^*))$, and $x^*_{s^*, u^{\tilde{a}}}$ is an optimal continuing allocation at $(s^*, u^{\tilde{a}})$. Like x^*, $x^{\tilde{a}}$ pools the initial interval $[0, s^*]$ and uses an optimal continuing allocation for higher types. The difference is that $x^{\tilde{a}}$ pools the initial interval on $\tilde{a}$ instead of a, i.e., $L^P(x^a) = L^P(x^*)$. Since x^* is optimal, and $a > 0$ is interior, it must be the case that $\left.\frac{d\, L^P(x^{\tilde{a}})}{d\tilde{a}}\right|_{\tilde{a}=a} = 0$. Evaluating and simplifying this derivative gives

$$(a - R(0, s^*))F([0, s^*]) + (a - s^*)V_{u^-}(s^*, u)F([s^*, M]) = 0. \tag{19}$$

However, since $s^* > 0$, $f(0) > 0$, and $V_{u^-} > 0$ by the alignment principle, (18) and (19) contradict each other. This contradicts the optimality of x^* proving the result in this case.

Now consider that Case 2 holds. There exists a pair of subsequences $t_n, s_n \in \tau_n(T)$ such that as $n \to \infty$, $t_n \to 0$, $s_n \to 0$, and $\frac{t_n}{s_n - t_n} \to 0$. By Lemma 10,

$$L^A(s_n | x_n) \geq \left(\sqrt{L^A(t_n | x_n) + \rho(s_n - t_n)} - (s_n - t_n)\right)^2 \equiv u_n.$$

By the alignment principle, this means that $L^P(x_n) \geq \int_0^{s_n}(x_n(t')-t')^2 f(t')dt' + F([s_n, M])V(s_n, u_n)$. Now construct an alternative sequence of allocations z_n defined by

$$z_n(t) \equiv \begin{cases} s_n & t < s_n \\ x^*_{s_n,\rho(s_n - R(0,s_n))}(t) & t \geq s_n \end{cases}.$$

The loss from z_n is $L^P(z_n) = \int_0^{s_n}(s_n - t')^2 f(t')dt' + F([s_n, M])V(s_n, \rho(s_n - R(0, s_n))$. Note that if $u_n > \rho(s_n - R(0, s_n))$

$$\begin{aligned}
&\frac{1}{s_n - t_n}\left(L^P(x_n) - L^P(z_n)\right) \\
\geq &\frac{1}{s_n - t_n}\left(\int_0^{s_n}\left((x_n(t') - t')^2 - (s_n - t')^2\right) f(t')dt' + F([s_n, M])(V(s_n, u_n) - V(s_n, \rho(s_n - R(0, s_n))))\right) \\
\geq &\frac{1}{s_n - t_n}\left(-s_n^2 F([0, s_n]) + F([s_n, M])k(u_n - \rho(s_n - \underline{\beta} s_n))\right) \\
\geq &\frac{1}{s_n - t_n}\left(-s_n^2 F([0, s_n]) + F([s_n, M])k\left(-2(s_n - t_n)\sqrt{L^A(t_n|x_n) + \rho(s_n - t_n)} + (s_n - t_n)^2 + \rho(\underline{\beta} s_n - t_n)\right)\right) \\
= &- s_n \frac{s_n}{s_n - t_n} F(0, s_n) + F([s_n, M])k\left(-2\sqrt{L^A(t_n|x_n) + \rho(s_n - t_n)} + (s_n - t_n) + \frac{\rho(\underline{\beta} s_n - t_n)}{s_n - t_n}\right).
\end{aligned}$$

The first inequality uses the alignment principle and the fact that $L^A(s_n|x_n) \geq u_n$. The second inequality uses the bound on the derivative in the alignment principle, and the definition of $\underline{\beta}$. The next inequality expands the definition of u_n and uses that $L^A(t_n|x_n) > 0$. Taking the limit as $n \to \infty$ gives,

$$\lim_{n\to\infty} -s_n \frac{s_n}{s_n - t_n} F(0, s_n) + F([s_n, M])k\left(2\sqrt{L^A(t_n|x_n) + \rho(s_n - t_n)} + (s_n - t_n) + \frac{\rho(\underline{\beta} s_n - t_n)}{s_n - t_n}\right) = \rho\underline{\beta}$$

This equality is due to (i) $L^A(t_n|x_n) \to 0$ because $t_n \to 0$, and (ii) that by hypothesis in case 2, $s_n \to 0$ and $\frac{t_n}{s_n - t_n} \to 0$. This contradicts the optimality of x_n which completes the proof in case 2 and thereby the full argument. *Q.E.D.*

D. Proofs from Section 4

D.1. Proof of Lemma 5

Proof. Note that $\overline{u}(t, u) = u + \rho t - 2t\sqrt{u + \rho R(t)} + t^2$. Since $\overline{u}(t, u)$ is decreasing in $R(t)$, it suffices to prove the lemma in the uniform limit where $R(t)$ is maximizes at $R(t) = t/2$ and

$\overline{u}_0(t,u) \equiv u + \rho t - 2t\sqrt{u+\rho t/2} + t^2$. Observe that $\overline{u}_0(t,u)$ is continuously differentiable in t $\forall u, t$. In addition, $\forall u$ $\overline{u}_0(0,u) - u = 0$. This means that in order to prove the lemma I only need to establish that $\overline{u}_0(t,u) - u$ is strictly single crossing from below in t for $u < \rho^2/4$, and that $\overline{u}_0(t,u) - \rho^2/4$ is strictly single crossing from below for $u \geq \rho^2/4$.

First suppose $u < \rho^2/4$. Assume $t > 0$ and $\overline{u}_0(t,u) = u$, i.e.

$$\rho t - 2t\sqrt{u+\rho t/2} + t^2 = 0. \tag{20}$$

I next evaluate the derivative of $\overline{u}_0(t,u)$ with respect to t.

$$\begin{aligned}
\frac{d(\overline{u}_0(t,u))}{dt} &= \rho - 2\sqrt{u+\rho t/2} + 2t - \frac{\rho t/2}{\sqrt{u+\rho t/2}} \\
&= t - t\left(\frac{\rho/2}{\sqrt{u+\rho t/2}}\right) \\
&= \frac{1}{2\sqrt{u+\rho t/2}}\left(2t\sqrt{u+\rho t/2} - \rho t\right) \\
&= \frac{t^2}{2\sqrt{u+\rho t/2}} > 0.
\end{aligned}$$

The second and last equality use (20). Note that for $t = 0$, $\frac{d(\overline{u}_0(t,u))}{dt} = \rho - 2\sqrt{u}$ which is positive if $u < \rho^2/4$.

Now suppose $u \geq \rho^2/4$.

$$\overline{u}(t,u) \geq \rho^2/4 \tag{21}$$

$$\iff \rho t - 2t\sqrt{u+\rho t/2} + t^2 + u > \rho^2/4. \tag{22}$$

note that the LHS of (22) is convex in u with a minimum at $u = \max\{\rho^2/4, t^2 - \rho t/2\}$. If $\max\{\rho^2/4, t^2 - \rho t/2\} = \rho^2/4 = u$, then (22) becomes,

$$\rho - 2\sqrt{\rho^2/4 + \rho t/2} + t > 0$$

which holds for all $t > 0$ as the LHS is strictly convex in t with a 0 derivative at $t = 0$. If instead $\max\{\rho^2/4, t^2 - \rho t/2\} = t^2 - \rho t/2 = u$, then the LHS of (22) evaluates to $\rho t/2 > 0$. This completes the proof. *Q.E.D.*

D.2. Proof of Proposition 2

Proof. I begin by proving the following approximation claim on the principal's and agent's loss for small intervals.

Claim 9. *Take any continuing allocation x at $u \geq 0$ with a threshold at $\delta \in \overline{\tau}(T)$ with $\delta < \overline{\delta}$. Let z be an alternative pooling continuing allocation with first threshold δ, i.e. it satisfies $z(t') = \sqrt{u + \rho R(\delta)}$ $\forall t' \in [0, \delta]$. It holds that*

$$\begin{aligned}
&\frac{\int_0^\delta \left((z(t') - t')^2 - (x(t') - t')^2\right) \lambda e^{-\lambda t'} dt'}{1 - e^{-\lambda\delta}}. \\
\leq &\max\{0, L^A(\delta|z) - L^A(\delta|x)\} && (23) \\
\leq &2\delta^{3/2}\sqrt{\rho} && (24)
\end{aligned}$$

Proof of Claim: Note that,

$$\begin{aligned}
&\frac{\int_0^\delta \left((z(t') - t')^2 - (x(t') - t')^2\right) \lambda e^{-\lambda t'} dt'}{1 - e^{-\lambda\delta}} \\
= &\frac{\int_0^\delta \left(L^A(t'|z) - L^A(t'|x)\right) \lambda e^{-\lambda t'} dt'}{1 - e^{-\lambda\delta}} \\
\leq &\max_{t' \leq \delta} L^A(t'|z) - L^A(t'|x),
\end{aligned}$$

where the last inequality is due to the Cauchy mean value theorem. By Lemma 8 the derivative of $L^A(t'|z) - L^A(t'|x)$ with respect to t' is given by $2(x(t') - z(t'))$. Since x is increasing and z is constant, the integrand is convex in t' and thereby maximized at one of its endpoints. Because both are continuing allocations at u, $L^A(0|z) - L^A(0|x) = 0$, which delivers the first inequality. By Lemma 10

$$\begin{aligned}
&L^A(\delta|z) - L^A(\delta|x) \\
\leq &\left(\sqrt{u + \rho R(\delta)} - \delta\right)^2 + \rho(\delta - R(\delta)) - \left(\sqrt{u + \rho\delta} - \delta\right)^2 \\
= &2\delta\left(\sqrt{u + \rho\delta} - \sqrt{u + \rho R(\delta)}\right) \\
\leq &2\delta^{3/2}\sqrt{\rho}.
\end{aligned}$$

The last inequality follows from concavity of the square root function. This proves the claim.

Suppose that for some $u \geq 0$, $L^P(y_u) > V(u) + 2\varepsilon$ for some small $\varepsilon > 0$. Let y^* be a continuing allocation at u that achieves $L^P(y^*) < V(u) + \varepsilon$. Suppose first that there is a highest threshold $t = \max\ \underline{\tau}_{y^*}(T)$ under y^*, i.e. y^* is pooling after t. Then note that

$$\begin{aligned}
&\frac{\int_t^\infty \lambda(y^*(t') - t')^2 e^{-\lambda t'} dt'}{e^{-\lambda t}} \\
=&\lim_{\bar{t}\to\infty} \int_0^{\bar{t}} \lambda(a(\bar{t}, u) - t')^2 e^{-\lambda t'} dt' + e^{-\lambda \bar{t}} L^P(y_{\overline{u}(\bar{t},u)}) \\
\geq & L^P(y_{L^A(t|y^*)})
\end{aligned}$$

where the equality uses $\frac{d\,L^P(y_u)}{du} \leq 1$ and the inequality is by Lemma 5 and the assumption that y_u satisfies the Bellman equation. In this case set $t_0 = t$.

Now suppose that $\underline{\tau}_{y^*}(T)$ is unbounded. Note that $y^*(0) < \sqrt{u + \frac{\rho}{\lambda}}$ in order to satisfy IC of type $t = 0$, Thus by IC for type t, it must be that $L^A(t|y^*) < (y^*(0) - t)^2 - r_{y^*}(0)$ which implies $L^A(t|y^*) < (\sqrt{u + \frac{\rho}{\lambda}} - t)^2$. But then, because $\frac{d\,L^P(y_u)}{du} \leq 1$ by assumption, there exists large enough $t \in \underline{\tau}(T)$, such that

$$\begin{aligned}
L^P(y_{L^A(t|y^*)}) <& L^P(y_{\underline{u}}) + L^A(t|y^*) - \underline{u} \\
<& L^P(y_{\underline{u}}) + (\sqrt{u + \frac{\rho}{\lambda}} - t)^2 \\
<& e^{\lambda t}\varepsilon/2.
\end{aligned}$$

Take such a t and set $t_0 = t$.

Take arbitrary $\delta \in (0, \bar{\delta})$. I will iteratively revise y^* at a sequence of thresholds decreasing from t_0 to 0 in finite steps and end with y_u. Define t_{i+1} recursively as follows.

Case 1: there exists a first threshold $s \in \underline{\tau}_{y^*}(T)$ such that $s \leq t_i - \delta$, i.e. $\underline{\tau}_{y^*}(t') = s\ \forall t' \in [s, t_i)$, in which case define $t_{i+1} \equiv s$.

Case 2: case 1 does not hold, in which case define $t_{i+1} = \min([t_i - \delta, t_i] \cap \underline{\tau}_{y^*}(T))$.

Note that one of the two cases must hold, and that in either case $t_i - t_{i+2} > \delta\ \forall i$. Now define,

$$y_i(t') \equiv \begin{cases} y^*(t') & t' < t_i \\ y_{L^A(t_i|y^*)}(t' - t_i) & t' \geq t_i \end{cases}.$$

By construction the loss from changing from y^* to y_0 is less than $\varepsilon/2$. I now analyze the change in loss from y_i to y_{i+1}. Consider the continuing allocation z at $L^A(t_{i+1}|y^*)$ defined by

$$z(t) \equiv \begin{cases} a(t_i - t_{i+1}, L^A(t_{i+1}|y^*)) & t' < t_i - t_{i+1} \\ y_{\overline{u}(t_i - t_{i+1}, L^A(t_{i+1}|y^*))}(t' - (t_i - t_{i+1})) & t' \geq t_i - t_{i+1} \end{cases}.$$

Allocation y_i uses y^* below t_i and the associated y_u above t_i. Allocation z pools t_{i+1} to t_i and then uses the appropriate y_u above t_i. The losses from z and y_i above t_{i+1} are given respectively by

$$\int_{t_{i+1}}^{t_i} \lambda \left(a(t_i - t_{i+1}, L^A(t_{i+1}|y^*)) - (t' - t_{i+1})\right)^2 e^{-\lambda t'} dt' + L^P(y_{\overline{u}(t_i - t_{i+1}, L^A(t_{i+1}|y^*))}) e^{-\lambda t_i}, \tag{25}$$

$$\int_{t_{i+1}}^{t_i} \lambda (y^*(t') - t')^2 e^{-\lambda t'} dt' + L^P(y_{L^A(t_i|y^*)}) e^{-\lambda t_i}. \tag{26}$$

If t_{i+1} is chosen according to case 1, then $z(t - t_{i+1}) = y_i(t)$ $\forall t \geq t_{i+1}$ and the two losses are equivalent. If case 2 holds instead, then $t_i - t_{i+1} < \delta$. Now note that the difference between (25) and (26) is

$$\begin{aligned}
&\frac{\int_0^{t_i - t_{i+1}} \left(\left(a(t_i - t_{i+1}, L^A(t_{i+1}|y^*)) - t'\right)^2 - (y^*(t' + t_{i+1}) - (t' + t_{i+1}))^2\right) \lambda e^{-\lambda t'} dt'}{1 - e^{-\lambda(t_i - t_{i+1})}} (e^{-\lambda t_{i+1}} - e^{-\lambda t_i}) \\
&+ \left(L^P(y_{\overline{u}(t_i - t_{i+1}, L^A(t_{i+1}|y^*))}) - L^P(y_{L^A(t_i|y^*)})\right) e^{-\lambda t_i} \\
\leq & 2\sqrt{\rho}(t_i - t_{i+1})^{3/2}(e^{-\lambda t_{i+1}} - e^{-\lambda t_i}) + 2\sqrt{\rho}(t_i - t_{i+1})^{3/2} e^{-\lambda t_i} \\
\leq & 2\sqrt{\rho}(t_i - t_{i+1})^{3/2} \\
\leq & 2\delta^{3/2}\sqrt{\rho}.
\end{aligned} \tag{27}$$

The first inequality applies Claim 9 twice and the fact that $\frac{d\, L^P(y_u)}{du} \leq 1$. The loss from y_{i+1} above t_{i+1} is $L^P(y_{L^A(t_{i+1}|y^*)}) e^{-\lambda t_{i+1}}$ which is less than that of z by Lemma 5 and the assumption that y_u satisfies the Bellman equation. Thus the expression in (27) is actually an upper bound on the difference in loss between y_{i+1} and y_i.

Note that because $t_i - t_{i+2} > \delta$ there exists $n < 2t_0/\delta$ such that $y_n = y_u$. Thus the total change in loss at the end of the process is bounded above by the sum over (27) given by

$$\begin{aligned}
& n\delta^{3/2}\sqrt{\rho} \\
\leq & 4t_0\delta^{1/2}\sqrt{\rho}
\end{aligned}$$

which goes to 0 with δ. Therefore, by choosing δ small enough, one can guarantee that this sum is less than $\varepsilon/2$, which contradicts the hypothesis that $L^P(y_u) > V(u) + 2\varepsilon$. *Q.E.D.*

D.3. Proof of Theorem 2

It will be convenient to notate the principal's separating loss as $L^P(d_u) \equiv V_s(u)$.

Proof. First I prove some claims about the first and second derivatives of $V_s(u)$.

Claim 10. *The separating allocation satisfies the following.*

1. $V_s'(u) = \begin{cases} \int_0^\infty \frac{\rho - 2\sqrt{D_u(t')}}{\rho - 2\sqrt{u}} \lambda e^{-\lambda t'} dt' & u \neq \rho^2/4 \\ \frac{\rho\lambda}{\rho\lambda + 2} & u = \rho^2/4 \end{cases}.$
2. $V_s'(u) = \lambda \frac{V_s(u) - u}{\rho - 2\sqrt{u}} \quad \forall u \neq \rho^2/4.$
3. $V_s'(u) \leq 1 \quad \forall u \geq 0.$

Proof of Claim: To see the first point, note that because of (2b), $\forall t \geq t_0 \geq 0$ and $u \geq 0$, $D_u(t) = D_{D_u(t_0)}(t - t_0)$. In words, starting the separating allocation at type 0 and initial loss u gives the same loss for type t as starting at type $t_0 > 0$ with initial loss given by that for type t under the original separating allocation, i.e., $D_u(t_0)$. We can implicitly differentiate this equation,

$$\begin{aligned} \frac{d(D_u(t))}{dt_0} &= \frac{d(D_{D_u(t_0)}(t - t_0))}{dt_0} \\ \iff 0 &= \frac{d(D_{\tilde{u}}(t - t_0))}{d\tilde{u}}\Big|_{\tilde{u} = D_u(t_0)} D_u'(t_0) - D'_{D_u(t_0)}(t - t_0) \\ \iff \frac{d(D_{\tilde{u}}(t - t_0))}{d\tilde{u}}\Big|_{\tilde{u} = D_u(t_0)} &= \frac{\rho - 2\sqrt{D_{D_u(t_0)}(t - t_0)}}{\rho - 2\sqrt{D_u(t_0)}} \end{aligned}$$

We can then evaluate the last line at $t_0 = 0$ to get

$$\frac{d(D_u(t))}{du} = \frac{\rho - 2\sqrt{D_u(t)}}{\rho - 2\sqrt{u}} \tag{28}$$

To see the second point, note that

$$\begin{aligned} D_u'(t) = \frac{d\big(d_u(t) - t)^2\big)}{dt} &= 2(d_u'(t) - 1)(d_u(t) - t) \\ =& 2\Big(\frac{\rho}{2(d_u(t) - t)} - 1\Big)(d_u(t) - t) = \rho - 2(d_u(t) - t) \\ =& \rho - 2\sqrt{D_u(t)}, \end{aligned} \tag{29}$$

where the first equality in the second line follows from (2b). Now write,

$$\begin{aligned} V_s(u) =& \int_0^\infty D_u(t') \lambda e^{-\lambda t'} dt' \\ =& u + \int_0^\infty \left(\rho - 2\sqrt{D_u(t')}\right) e^{-\lambda t'} dt'. \end{aligned}$$

The first line is the definition of the separating loss for the principal, while the second line uses integration by parts and (29). Using the first point in Theorem 2 for $u \neq \rho^2/4$ in conjunction with this equation gives point 2.

The third point in the claim follows from inspecting the integrand in the first point and using the fact in Lemma 2 that $\rho - 2\sqrt{u} \geq (\leq) 0 \implies \rho^2/4 \geq (\leq) D_u(t) \geq (\leq) u$.

Claim 11. *The separating allocation satisfies the following.*

1. $V_s''(u) = \begin{cases} \int_0^\infty \frac{\rho - 2\sqrt{D_u(t')}}{(\rho-2\sqrt{u})^2} \frac{2\left(\sqrt{D_u(t')} - \sqrt{u}\right)}{\sqrt{u}\sqrt{D_u(t')}} \lambda e^{-\lambda t'} dt' & u \neq \rho^2/4 \\ \frac{4\lambda\rho}{(\lambda\rho+2)(\lambda\rho+4)} & u = \rho^2/4 \end{cases}.$
2. $V_s''(u) = \frac{V_s'(u)(\lambda + \frac{1}{\sqrt{u}}) - \lambda}{\rho - 2\sqrt{u}} \quad \forall u \neq \rho^2/4$
3. $V_s''(u) \geq 0 \quad \forall u \geq 0$

Proof of Claim: To see point 1, take the derivative of the expression in point 1 of Claim 10 in u. This uses the fact (established in the proof of that point) that $\frac{d(D_u(t))}{du} = \frac{\rho - 2\sqrt{D_u(t)}}{\rho - 2\sqrt{u}}$. To see point 2, take the derivative of both the left hand side and right hand side of point 2 of Claim 10. To get point 1 for the case of $u = \rho^2/4$, use l'Hopital's rule on the expression in point 2. Point 3 is seen by inspecting point 1 and noting that $\rho - 2\sqrt{u} \geq (\leq) 0 \implies \rho^2/4 \geq (\leq) D_u(t) \geq (\leq) u$.

I will confirm that

$$V_s(u) \leq \int_0^t \lambda(a(t,u) - t')^2 e^{-\lambda t'} dt' + e^{-\lambda t} V_s(\overline{u}(t,u)) \quad \forall t > 0, \ \forall u \geq \rho^2/16. \tag{30}$$

I will first show that there exists a large λ such that the condition holds for all $\lambda' > \lambda$. I will then show that this condition is tighter for higher λ, i.e. if it holds for some fixed λ then it holds for all lower λ. Define

$$\tilde{V}(t,u) \equiv \int_0^t \lambda(a(t,u) - t')^2 e^{-\lambda t'} dt' + e^{-\lambda t} V_s(\overline{u}(t,u)) \tag{31}$$

as the principal's loss from choosing first threshold t and separating thereafter given initial loss u.

Part 1: For any $u > \rho^2/16$. There exists a large $\tilde{\lambda}$ such that $\forall \lambda > \tilde{\lambda}$, $\forall t > 0$, $\tilde{V}(t,u) - V_s(u) > 0$ holds.

More specifically, for the same qualifiers, I will show that $\frac{d\,\tilde{V}(t,u)}{dt} \geq 0 \ \forall t > 0$. Since $\tilde{V}(0,u) = V_s(u)$, this completes part 1. I begin by expanding and simplifying this derivative condition.

$$
\begin{aligned}
&\frac{d\left(\int_0^t \lambda(a(t,u)-t')^2 e^{-\lambda t'}dt' + e^{-\lambda t}V_s(\overline{u}(t,u))\right)}{dt} \\
=&\rho R'(t)\frac{(a(t,u)-R(t))(1-e^{-\lambda t})}{a(t,u)} + \left((a(t,u)-t)^2 - V_s(\overline{u}(t,u))\right)\lambda e^{-\lambda t} \\
&+ e^{-\lambda t}V_s'(\overline{u}(t,u))\overline{u}_t(t,u) \geq 0 \\
\iff& \rho\lambda e^{-\lambda t}(t-R(t))\frac{(a(t,u)-R(t))}{a(t,u)} + \left(\overline{u}(t,u) - \rho(t-R(t)) - V_s(\overline{u}(t,u))\right)\lambda e^{-\lambda t} \\
&+ e^{-\lambda t}V_s'(\overline{u}(t,u))\overline{u}_t(t,u) \geq 0 \\
\iff& -\rho(t-R(t))\frac{R(t)}{a(t,u)} + \left(\overline{u}(t,u) - V_s(\overline{u}(t,u))\right) \\
&+ V_s'(\overline{u}(t,u))\overline{u}_t(t,u)/\lambda \geq 0,
\end{aligned}
$$

where the second equality uses $R'(t) = (t-R(t))\frac{\lambda e^{-\lambda t}}{1-e^{-\lambda t}}$ and $\overline{u}(t,u) = (a(t,u)-t)^2 + \rho(t-R(t))$. Using Claim 10 gives,

$$
V_s'(\overline{u}(t,u))\frac{d}{dt}\overline{u}(t,u) = \lambda\frac{V_s(\overline{u}(t,u)) - \overline{u}(t,u)}{\rho - 2\sqrt{\overline{u}(t,u)}}\left(\rho - \rho\frac{tR'(t)}{a(t,u)} - 2(a(t,u)-t)\right).
$$

Plugging the above identity to the above inequality gives

$$
\begin{aligned}
&-\rho(t-R(t))\frac{R(t)}{a(t,u)} + \frac{V_s(\overline{u}(t,u)) - \overline{u}(t,u)}{\rho - 2\sqrt{\overline{u}(t,u)}}\left(-\rho\frac{tR'(t)}{a(t,u)} + 2(\sqrt{\overline{u}(t,u)} + t - a(t,u))\right) \geq 0 \\
\iff& -R(t) + \frac{V_s(\overline{u}(t,u)) - \overline{u}(t,u)}{\rho - 2\sqrt{\overline{u}(t,u)}}\left(\frac{2a(t,u)}{a(t,u) + \sqrt{\overline{u}(t,u)} - t} - \frac{t\lambda e^{-\lambda t}}{1-e^{-\lambda t}}\right) \geq 0.
\end{aligned}
$$

The last implication uses the identity, from IC of the threshold type t, that $(\sqrt{\overline{u}(t,u)} + t - a(t,u))(\sqrt{\overline{u}(t,u)} + a(t,u) - t) = \rho(t - R(t))$. Reusing this identity and dividing both sides by $R(t)$ reduces the above inequality to,

$$
\frac{V_s(d_1^2) - d_1^2}{\rho - 2d_1}\left(\frac{\frac{t}{R(t)}(2(d_0+d_1)-\rho)+\rho}{(d_0+d_1)^2} + \lambda\right) \geq 1. \tag{32}
$$

Recall the definition of $d_1 = \sqrt{\overline{u}(t,u)}$ and $d_0 = a(t,u) - t$. We can multiply both sides by λ and use the definition of $V_s'(\cdot)$ in Claim 10 to get that (32) holds if and only if

$$\frac{\frac{t}{R(t)}(2(d_0 + d_1) - \rho) + \rho}{(d_0 + d_1)^2} \geq \frac{\lambda(1 - V_s'(d_1^2))}{V_s'(d_1^2)} \tag{33}$$

First consider the RHS of (33).

Claim 12. $\forall \varepsilon > 0$, $\exists \lambda_0$ *such that* $\forall \lambda > \lambda_0$, $\forall t > 0$, $\frac{\lambda(1-V_s'(d_1^2))}{V_s'(d_1^2)} < \frac{1}{d_1} + \varepsilon$.

Proof of Claim: Write

$$V_s'(d_1^2) = \int_0^\infty \frac{\rho - 2\sqrt{D_{d_1^2}(t')}}{\rho - 2d_1} \lambda e^{-\lambda t'} dt'.$$

Note that this implies that $V_s'(d_1^2) \to 1$ as $\lambda \to \infty$. Also note that because Claim 11 $V_s'(\cdot)$ is increasing, and by Lemma 5 $d_1^2 = \overline{u}(t,u) > u = \overline{u}(0,u)$, this convergence is uniform across $t > 0$. Now use the same identity for $V_s'(u)$ to rewrite

$$\begin{aligned} \lambda(1 - V_s'(d_1^2)) =& \lambda \int_0^\infty \frac{2\left(\sqrt{D_{d_1^2}(t')} - d_1\right)}{\rho - 2d_1} \lambda e^{-\lambda t'} dt' \\ <& \lambda \frac{2\left(\sqrt{D_{d_1^2}(1/\lambda)} - d_1\right)}{\rho - 2d_1} \end{aligned} \tag{34}$$

The first inequality uses Jensen's inequality and the fact that $\frac{\sqrt{D_{d_1^2}(t')}}{\rho - 2d_1}$ is concave in t'. This last point follows from the fact that $D_u'(t') = \rho - 2\sqrt{D_u(t')}$ and $u < (>)((=))D_u(t') < (>)((=))\rho^2/4$ by Lemma 2. We will show that this expression converges pointwise to $1/d_1$. Note that $\frac{2(\sqrt{D_{d_1^2}(1/\lambda)} - d_1)}{\rho - 2d_1}$ is decreasing in d_1 which will be useful for uniform convergence—its derivative in d_1 is

$$-2d_1 \Big(\frac{\rho - 2\sqrt{D_{d_1^2}(t')}}{(\rho - 2d_1)^2} \Big) \Big(\frac{2\left(\sqrt{D_{d_1^2}(t')} - d_1\right)}{d_1 \sqrt{D_{d_1^2}(t')}} \Big),$$

which is negative again because $u < (>)((=))D_u(t') < (>)((=))\rho^2/4$ by Lemma 2. In addition $1/d_1$ is continuous in d_1 and lies in the compact set $[0, 1/\sqrt{u}]$. Indeed using a change of variable to $\gamma \equiv 1/d_1$, We have pointwise convergence of a monotone function in γ to a continuous function γ on the compact set $[0, 1/\sqrt{u}]$ and so this convergence is uniform.

To see that (34) converges pointwise to $1/d_1$, as $\lambda \to \infty$, use l'Hopitals rule as follows. So

$$\lim_{\lambda\to\infty} \frac{\frac{2(\sqrt{D_{d_1^2}(1/\lambda)}-d_1)}{\rho-2d_1}}{1/\lambda} = \lim_{k\to 0} \frac{\frac{2(\sqrt{D_{d_1^2}(k)}-d_1)}{\rho-2d_1}}{k}$$
$$= \lim_{k\to 0} \frac{\rho - 2\sqrt{D_{d_1^2}(k)}}{(\rho-2d_1)\sqrt{D_{d_1^2}(k)}} = \frac{1}{d_1}.$$

This proves the claim.

Given the claim above, establishing (33) reduces to finding $\varepsilon > 0$ and $\tilde{\lambda}$ such that $\forall t > 0$ and $\forall \lambda > \tilde{\lambda}$.

$$\frac{\frac{t}{R(t)}(2(d_0+d_1)-\rho)+\rho}{(d_0+d_1)^2} \geq \frac{1}{d_1} + \varepsilon \tag{35}$$

First we show the following helpful inequalities. Define $L_{u,\lambda} \equiv \{t : \; R(t) + \rho/4 < \sqrt{u+\rho R(t)}\}$. It will be convenient to notate $d_1 \equiv \sqrt{\overline{u}(t,u)}$ and $d_0 \equiv a(t,u) - t$

Lemma 15. *The following hold.*

1. $t \in L_{u,\lambda} \implies d_1 + d_0 > \rho/2$. *Moreover, for* $\rho\lambda > 2$ *and* $\forall u \geq \rho^2/16$, $L_{u,\lambda} = [0,\infty)$.
2. $\forall t, \lambda > 0$, $\underline{u} < \tilde{u} < \rho^2/4$, $\min\{\rho/4, 2\sqrt{\underline{u}}\}, < d_1 + d_0 < \rho$.
3. $\forall u > 0$, $\exists \bar{t}_1$ *such that* $\forall t \geq \bar{t}_1$ $d_1 + d_0 < \rho$.

Proof. Suppose instead that $d_1 + d_0 \leq \rho/2$. By definition $(d_1+d_0)(d_1-d_0) = \rho(t - R(t))$. Using the assumed inequality gives

$$\rho/2(\rho/2 - 2d_0) \geq \rho(t - R(t))$$
$$\iff \rho/2 - 2(a(t,u) - t) \geq 2(t - R(t))$$
$$\iff \rho/4 + R(t) \geq \sqrt{u + \rho R(t)}$$
$$\implies t \notin L_{u,\lambda}.$$

Next, if $\rho\lambda > 2$, then because $R(t) < 1/\lambda$ we have $R(t) < \rho/2$ which implies that $\forall u \geq \rho^2/16$, $L_{u,\lambda} = [0,\infty)$.

To see the second part of the lemma bounding $d_1 + d_0$ from above, first note that $d_0 < \rho/2$. To see this expand $d_0 = \sqrt{u + \rho R(t)} - t < \sqrt{\rho^2/4 + \rho t/2} - t$. And so the inequality holds if

$\sqrt{\rho^2/4 + \rho t/2} \leq t + \rho/2 \iff 0 \leq \rho t/2 + t^2.$

$$\begin{aligned} d_1 + d_0 < \rho &\iff d_1^2 < \rho^2 + d_0^2 - 2\rho d_0 \iff t - R(t) < \rho - 2d_0 \\ &\Longleftarrow t/2 < \rho - 2\sqrt{\rho^2/4 + \rho t/2} + 2t \iff 0 < \rho t + 9t^2/4. \end{aligned}$$

Both of these arguments use that $R(t) < t/2$ and $u \leq \rho^2/4$.

To see the second part of the lemma bounding $d_1 + d_0$ from below, take $k > 0$. First note that if $d_0 > k$, then $d_1 + d_0 > k$ because $d_1 \geq 0$. So assume $d_0 \leq k$ and observe that

$$\begin{aligned} d_0 + d_1 > k &\iff d_1 > k - d_0 \\ &\iff \rho(t - R(t)) - k^2 + 2kd_0 > 0 \end{aligned}$$

Since d_0 is strictly increasing in u, it is without loss to establish the weak inequality for $u = \underline{u}$.Note that the LHS is strictly quasi-concave in $R(t)$, indeed its derivative is $\rho(-1 + \frac{k}{\sqrt{\underline{u} + \rho R(t)}}$ which if positive is positive for a lower $R(t)$; and if negative is negative for a higher $R(t)$. Since $0 < R(t) < t/2 \; \forall t > 0$, one only needs to establish the inequality for $R(t) = t/2$ and $R(t) = 0$ respectively given by

$$\begin{aligned} \rho t/2 - k^2 + 2k\sqrt{\underline{u} + \rho t/2} - 2kt > 0 \\ \rho t - k^2 + 2k\underline{u} - 2kt > 0. \end{aligned}$$

Notice that both inequalities hold at $t = 0$ if $k \leq 2\sqrt{\underline{u}}$. Moreover both LHS's are strictly increasing in t $\forall t > 0$ if $k \leq \rho/4$ proving that $d_1 + d_0 > \min\{2\sqrt{\underline{u}}, \rho/4\}$.

To see the last part of the lemma take $\bar{t}_0$ large enough such that $t > \bar{t}_0$ implies $\sqrt{u + \rho R(t)} < t$. note that because of the concavity of the square root function,

$$\begin{aligned} d_1 + d_0 =& \sqrt{u + \rho R(t)} - t + \sqrt{(\sqrt{u + \rho R(t)} - t)^2 + \rho(t - R(t))} \\ <& \frac{\rho(t - R(t))}{2(t - \sqrt{u + \rho R(t)})} \end{aligned}$$

Since $R(t) \in [0, 1/\lambda_1]$ because $\lambda > \lambda_1$, by taking $\bar{t}_1$ large, the final expression is less than ρ for $t > \bar{t}_1$. Thus taking $\bar{t} = \max\{\bar{t}_1, \bar{t}_0\}$ delivers the second part of the lemma.

Q.E.D.

We first show that for any initial compact interval of types there exists a large enough λ such that holds.

Claim 13. *$\forall \bar{t} > 0$ $\exists \lambda_2$ and $\varepsilon > 0$ such that $\forall \lambda > \lambda_0$ $t \in [0, \bar{t}]$, (D.3) holds.*

Proof of Claim: First note that because the exponential density is decreasing, $R(t) < t/2$ $\forall t, \lambda > 0$. Because $2(d_1 + d_0) > \rho$ by Lemma 15, it is sufficient to prove for the same qualifiers that

$$\frac{4(d_1 + d_0) - \rho}{(d_1 + d_0)^2} - \frac{1}{d_1} > \varepsilon \tag{36}$$

Notice that λ only affects this expression through $R(t)$. The derivative of the LHS in $R \equiv R(t)$ is given by

$$\frac{2\rho - 4(d_1 + d_0)}{(d_1 + d_0)^3} \frac{\rho}{2\sqrt{u + \rho R(t)}}(1 - \frac{t}{d_1}) - \frac{\rho}{2t\sqrt{u + R(t)}d_1}.$$

For $t \in [0, \bar{t}]$ for any $\bar{t}$, $d_1 + d_0$ lies in a compact interval and is uniformly bounded away from 0 by Lemma 15. Similarly $d_1 = \overline{u}(t, u) \geq u > \rho^2/16$. Lastly, $R(t) \in [0, 1/\lambda_2]$, for all t, so these derivatives are uniformly bounded below $\forall t \in [0, \bar{t}]$ and $\lambda > \lambda_2$ by some ν. Thus, because $R(t) \in [0, 1/\lambda]$ the LHS of (36) is within $-\nu/\lambda$ of this LHS evaluated at $R = 0$. Let $\underline{d}_0 \equiv \sqrt{u} - t$ and $\underline{d}_1 \equiv \sqrt{(\sqrt{u} - t)^2 + \rho t}$ be d_0 and d_1 evaluated at $R = 0$. It is sufficient to prove that,

$$\frac{4(\underline{d}_1 + \underline{d}_0) - \rho}{(\underline{d}_1 + \underline{d}_0)^2} - \frac{1}{\underline{d}_1} > 0 \; \forall t \in [0, \bar{t}]. \tag{37}$$

Then, continuity of the LHS in t gives that this LHS$> \varepsilon$ for some small $\tilde{\varepsilon} > 0$ for all $t \in [0, \bar{t}]$. Then taking λ_2 large so that $\tilde{\varepsilon} - \nu/\lambda > \varepsilon > 0$, $\forall \lambda > \lambda_2$ proves the claim. By multiplying by $(\underline{d}_1 + \underline{d}_0)^2 d_1$, (37) is equivalent to

$$\begin{aligned} & 3\underline{d}_1^2 - \underline{d}_0^2 + 2\underline{d}_1\underline{d}_0 > \rho d_1 \\ \Longleftarrow\ & 2\underline{d}_1^2 + 2\underline{d}_1\underline{d}_0 > \rho \underline{d}_1 \\ \Longleftrightarrow\ & \underline{d}_1 + \underline{d}_0 > \rho/2. \end{aligned}$$

The last inequality holds by Lemma 15. To see the $\Longleftarrow$ note that

$$3\underline{d}_1^2 - \underline{d}_0^2 = 3(\sqrt{u} - t)^2 + 3\rho t - (\sqrt{u} - t)^2 = 2(\sqrt{u} - t)^2 + 3\rho t = 2\underline{d}_1^2 + \rho t \geq 2\underline{d}_1^2.$$

Claim 14. *There exists $\varepsilon > 0$ and $\bar{t} > 0$ such that $\forall t > \bar{t}$ and $\lambda > \lambda_1$ (36) holds.*

Proof of Claim: Note that by Lemma 15 $\forall t > \bar{t}_0$ and $\lambda > \lambda_1$, starting with (36)

$$\begin{aligned}
&\frac{4(d_1+d_0)-\rho}{(d_1+d_0)^2} - \frac{1}{d_1} > \varepsilon \\
\iff &\frac{2(2(d_1+d_0)-\rho)}{(d_1+d_0)^2} + \frac{\rho}{(d_1+d_0)^2} - \frac{1}{d_1} > \varepsilon \\
\Longleftarrow &\frac{1}{\rho} - \frac{1}{d_1} > \varepsilon \qquad (38)
\end{aligned}$$

The $\Longleftarrow$ uses both inequalities in the Lemma 15 that $2(d_1+d_0)-\rho > 0$ and $d_1 + d_0 < \rho$. Note that for $t > \sqrt{u+1/\lambda}$ and because $t > R(t)$,

$$d_1 = \sqrt{(\sqrt{u+\rho R(t)}-t)^2 + \rho(t-R(t)} > t - \sqrt{u+\rho/\lambda_1}$$

$\forall \lambda > \lambda_1$. Thus take $\bar{t}$ such that for $t > \bar{t}$, $\frac{1}{t-\sqrt{u+\rho/\lambda_1}} < \frac{1}{2\rho}$ and $\varepsilon < \frac{1}{2\rho}$ establishes (38) and thereby (32) proving the claim.

Taking $\lambda > \lambda_i$ for $i = 0, 1, 2$ completes part 1.

Part 2: If for some $\tilde{\lambda}$, $\tilde{V}(t,u) - V_s(u) > 0$ $\forall u > \rho^2/16$ and $\forall t \in L_{u,\tilde{\lambda}}$, then $\forall \lambda' < \tilde{\lambda}$, $\tilde{V}(t,u) - V_s(u) > 0$ $\forall t \in L_{u,\lambda'}$ and $\forall u > \rho^2/16$.

Recall the definition, $L_{u,\lambda} \equiv \left\{t : R(t) + \rho/4 < \sqrt{u+\rho R(t)}\right\}$. Note that $\forall u \geq \rho^2/16$, $L_{u,\lambda} = [0, \bar{t}_{u,\lambda})$, where $\bar{t}_{u,\lambda}$ is increasing in λ, u.

Suppose the conclusion of part 2 is false, and let $\bar{\lambda}$ be the highest witness to this contradiction less than $\tilde{\lambda}$ with associated $\bar{t}$ so that $\tilde{V}(\bar{t},u) - V_s(u) \not> 0$. This exists because $\tilde{V}$ is differentiable in t, λ. Therefore, also by differentiability of $\tilde{V}$ in t and λ, the following conditions are satisfied at $\bar{\lambda}$ and $\bar{t}$,

$$\tilde{V}(\bar{t},u) - V_s(u) = 0, \text{ and} \qquad (39)$$

$$\frac{d\,\tilde{V}(\bar{t},u)}{dt} = 0. \qquad (40)$$

If (39) did not hold, then a slightly higher λ would also violate the condition. If (40) did not hold then because (39) holds, some nearby t to $\bar{t}$ would have $\tilde{V}(t,u) - V_s(u) < 0$, and then by the same logic one can find a higher λ that also violates the condition. This uses the fact that $t_{u,\lambda}$ is increasing in λ.

It will be helpful to explicitly notate λ in $R(t) \equiv R(t,\lambda)$. Let $k > 0$, and $y \equiv \sqrt{\rho R(\bar{t},k) + u}$ and $\overline{u} \equiv (y-t)^2 + \rho(s - R(\bar{t},k))$ be the alternative first action and first threshold loss under

$k = \lambda$. Define the continuing allocation

$$x(t') \equiv \begin{cases} y & t' < \bar{t} \\ d_{\overline{u}}(t' - \bar{t}) + \bar{t} & t' > \bar{t} \end{cases}.$$

Note that for $k = \overline{\lambda}$, $\tilde{V}(\bar{t}, u) - V_s(u) = \int_0^\infty (L^A(t'|x) - D_u(t'))\lambda e^{-\lambda t'} dt'$. Thus,

$$\begin{aligned} & \frac{d(\tilde{V}(\bar{t}, u) - V_s(u))}{d\lambda} && (41) \\ = & \frac{d(\int_0^\infty (L^A(t'|x) - D_u(t'))\lambda e^{-\lambda t})}{d\lambda}\Big|_{\lambda=k=\overline{\lambda}} + \frac{d(\int_0^\infty (L^A(t'|x) - D_u(t')))\overline{\lambda} e^{-\overline{\lambda} t})}{dk}\Big|_{k=\lambda=\overline{\lambda}} \\ = & \frac{d(\int_0^\infty (L^A(t'|x) - D_u(t')))\lambda e^{-\lambda t})}{d\lambda}\Big|_{\lambda=k=\overline{\lambda}} + \int_0^\infty \frac{d\ L^A(t'|x)}{dk}\Big|_{k=\lambda=\overline{\lambda}} \overline{\lambda} e^{-\overline{\lambda} t} && (42) \end{aligned}$$

The rest of the proof of part 2 shows that (41) is negative. Because of (39), this contradicts the fact that $\overline{\lambda}$ is the highest witness to a violation of $\tilde{V}(\bar{t}, u) - V_s(u) > 0$ below $\tilde{\lambda}$.

Step 1: The first term in (42) is negative.

Claim 15. *$L^A(t|x) - D_u(t)$ is single crossing from below in $t > 0$.*

Proof of Claim:

$$\frac{d\left(L^A(t|x) - D_u(t)\right)}{dt} = 2(d_u(t) - x(t)).$$

Since $d_u(t)$ is increasing, and $x(t)$ is constant for $t < \bar{t}$, $L^A(t|x) - D_u(t)$ is strictly convex in t for $t < \bar{t}$. Also $d_u(0) - x(0) < 0$ and $L^A(0|x) - D_u(0) = 0$, so $L^A(t|x) - D_u(t)$ is single crossing from below in t for $\bar{t} > t > 0$. Since $L^A(t|x) = D_{\overline{u}}(t - \bar{t})\ \forall t \geq \bar{t}$, Lemma 2 point 3 implies $\text{sign}(L^A(t|x) - D_u(t)) = \text{sign}(\overline{u} - D_u(\bar{t}))$. Continuity of $L^A(t|x)$ in t completes the proof of the claim.

This means that the first term in (42) is of the form $\mathbb{E}[g(t)|t \sim exp(\lambda)]$, where the function g is single crossing from below in t and independent of λ. Note also that an increase in λ corresponds to a downward monotone likelihood ratio shift in the exponential distribution, i.e. for $\lambda' < \lambda''$, $\frac{\lambda' e^{-\lambda' t}}{\lambda'' e^{-\lambda'' t}}$ is increasing in t. Theorem 2 from Athey (2002) delivers that the single crossing property is preserved under monotone likelihood ratio shifts. Because of (39) $\overline{\lambda}$, represents a crossing in this expectation and so its derivative in λ must be negative in order to satisfy single crossing. This completes step 1.

Step 2: The second term in (42) is negative.

First, I rewrite condition (40)

$$\begin{aligned}
&\frac{d}{dt}\left(\int_0^t \overline{\lambda}(a(t,u)-t')^2 e^{-\overline{\lambda}t'}dt' + e^{-\overline{\lambda}t}V_s(\overline{u}(t,u))\right)\Big|_{t=\overline{t}} \\
=&\frac{\rho R_t(\overline{t},k)}{a(\overline{t},u)}((a(\overline{t},u)-R(\overline{t},k))(1-e^{-\overline{\lambda}\overline{t}}) + (a(\overline{t},u)-\overline{t})V_s'(\overline{u}(\overline{t},u))e^{-\overline{\lambda}\overline{t}}) \\
&+((a(\overline{t},u)-\overline{t})^2 - V_s(\overline{u}(\overline{t},u)))\overline{\lambda}e^{-\overline{\lambda}\overline{t}} + V_s'(\overline{u}(\overline{t},u))(\rho - 2(a(\overline{t},u)-\overline{t})) = 0
\end{aligned}$$

I next note that if the second part of the last equality is negative then this completes this gives the desired inequality; that is,

$$((a(\overline{t},u)-\overline{t})^2 - V_s(\overline{u}(\overline{t},u)))\overline{\lambda}e^{-\overline{\lambda}\overline{t}} + V_s'(\overline{u}(\overline{t},u))(\rho - 2(a(\overline{t},u)-t))e^{-\overline{\lambda}\overline{t}}) < 0 \tag{43}$$

$$\begin{aligned}
&\implies ((a(\overline{t},u)-R(\overline{t},k))(1-e^{-\overline{\lambda}\overline{t}}) + (a(\overline{t},u)-\overline{t})V_s'(\overline{u}(\overline{t},u))e^{-\overline{\lambda}\overline{t}}) > 0 \\
&\iff 0 > \frac{\rho R_k(\overline{t},k)}{a(\overline{t},u)}((a(\overline{t},u)-R(\overline{t},k))(1-e^{-\overline{\lambda}\overline{t}}) + (a(\overline{t},u)-\overline{t})V_s'(\overline{u}(\overline{t},u))e^{-\overline{\lambda}\overline{t}}) \\
&\iff \int_0^\infty \frac{d\ L^A(t'|x)}{dk}\Big|_{k=\lambda=\overline{\lambda}}\overline{\lambda}e^{-\overline{\lambda}t} < 0.
\end{aligned}$$

The penultimate equivalence follows from $R_k(s,k) \leq 0$. The last expression is the second term in (42). Thus, establishing (43) is sufficient to complete the proof of step 2, and thereby part 2. I now simplify (43),

$$\begin{aligned}
&\left((a(\overline{t},u)-\overline{t})^2 - V_s(\overline{u}(\overline{t},u))\right)\overline{\lambda}e^{-\overline{\lambda}\overline{t}} + V_s'(\overline{u}(\overline{t},u))(\rho - 2(a(\overline{t},u)-t))e^{-\overline{\lambda}\overline{t}}) < 0 \\
\iff &\left(\overline{u}(\overline{t},u) - \rho(\overline{t}-R(\overline{t},k)) - V_s(\overline{u}(\overline{t},u))\right)\overline{\lambda}e^{-\overline{\lambda}\overline{t}} + \frac{V_s(\overline{u}(\overline{t},u)) - \overline{u}(t,u)}{\rho - 2\sqrt{\overline{u}(\overline{t},u)}}(\rho - 2(a(\overline{t},u)-\overline{t}))\overline{\lambda}e^{-\overline{\lambda}\overline{t}} < 0 \\
\iff &\frac{V_s(\overline{u}(\overline{t},u)) - \overline{u}(\overline{t},u)}{\rho - 2\sqrt{\overline{u}(\overline{t},u)}}\,2\left(\sqrt{\overline{u}(\overline{t},u)} - (a(\overline{s},u)-\overline{t})\right) < \rho(\overline{t}-R(\overline{t},k)) \\
\iff &\frac{V_s(\overline{u}(\overline{t},u)) - \overline{u}(s,u)}{\rho - 2\sqrt{\overline{u}(\overline{t},u)}}\,2\left(\sqrt{\overline{u}(\overline{t},u)} - (a(\overline{t},u)-\overline{t})\right) < \rho(\overline{t}-R(\overline{t},k)) \\
\iff &\frac{V_s(d_1^2) - d_1^2}{\rho - 2d_1}\frac{2}{d_0+d_1} < 1
\end{aligned}$$

The first equivalence uses the expression for $V_s'(\cdot)$ in Claim 10. Recall that $d_1 \equiv \sqrt{\overline{u}(t,u)}$ and $d_0 \equiv a(t,u)-t$. The last equivalence is due to the identity $(d_1+d_0)(d_1-d_0) = \rho(t-R(t))$ from the IC constraint of type t. But note that the simplified version of (40) in (32) means that it suffices to prove that,

$$\begin{aligned} & \frac{2}{d_0+d_1} < \frac{\frac{\bar{t}}{R(\bar{t})}(2(d_0+d_1)-\rho)+\rho}{(d_0+d_1)^2} + \bar{\lambda} \\ \Longleftarrow\ & 2(d_0+d_1) \geq \rho, \end{aligned}$$

where the $\Longleftarrow$ follows from $t/R(t) \geq 2$. Since $\bar{t} \in L_{u,\bar{\lambda}}$, this is given by Lemma 15. This completes part 2.

Part 3: $\forall \lambda > 0$, $\forall u \geq \rho^2/16$, and $t \notin L_{u,\lambda}$, $\tilde{V}(t,u) > V_s(u)$.

Let $u \geq \rho^2/16$, $t \notin L_{u,\lambda}$, and $0 \leq r \leq t$. Define the continuing allocation x_r at u by

$$x_r(t') \equiv \begin{cases} d_u(t') & t' < r \\ a(t-r, D_u(r)) + r & r \leq t' < t \\ d_{\overline{u}(t-r,D_u(r))}(t'-t) + t & t' > t \end{cases}$$

That is x_r separates on $[0,r)$, pools on $[r,t)$, and separates again on (t,∞). Note that $L^P(x_0) = \tilde{V}(t,u)$. Define $t - r \equiv \tilde{t}$.

Claim 16.

$$\frac{d\, L^P(x_r)}{dr} \geq 0 \implies \tilde{t} \in L_{D_u(r),\lambda}.$$

Proof of Claim: Notice that $L_{u,\lambda} = [0,\infty)$ for $u \geq \rho^2/16$, and $\lambda \geq \rho/2$. Since $D_u(r) > \rho^2/16$ because $u \geq \rho^2/16$, and Lemma 5, we can focus on the case in which $\rho\lambda < 2$.

By expanding and simplifying the relevant derivative in r we get,

$$
\begin{aligned}
&\frac{d\,L^P(x_r)}{dr} \geq 0 && (44)\\
\iff &\left(\frac{2(a(\tilde{t},D_u(r))-\sqrt{D_u(r)})(a(\tilde{t},D_u(r))-R(\tilde{t}))-\rho R(\tilde{t})(1-R'(\tilde{t}))}{a(\tilde{t},D_u(r))}+\rho(1-R'(\tilde{t}))\right)(1-e^{-\lambda\tilde{t}})\\
&-\rho\lambda R(\tilde{t})+\left(\frac{2(a(\tilde{t},D_u(r))-\sqrt{D_u(r)})(a(\tilde{t},D_u(r))-\tilde{t})-\tilde{t}\rho(1-R'(\tilde{t}))}{a(\tilde{t},D_u(r))}\right)V_s'(\overline{u}(\tilde{t},D_u(r)))e^{-\lambda\tilde{t}} \geq 0\\
\iff &\frac{2(a(\tilde{t},D_u(r))-R(\tilde{t}))}{a(\tilde{t},D_u(r))+\sqrt{D_u(r)}}(1-e^{-\lambda\tilde{t}})-\lambda R(\tilde{t})\\
&+\left(\frac{2(a(\tilde{t},D_u(r))-t)}{a(\tilde{t},D_u(r))+\sqrt{D_u(r)}}-\frac{\lambda t}{1-e^{-\lambda\tilde{t}}}\right)V_s'(\overline{u}(\tilde{t},D_u(r)))e^{-\lambda\tilde{t}} \geq 0\\
\iff &\left(\frac{2(a(\tilde{t},D_u(r))-R(\tilde{t}))}{a(\tilde{t},D_u(r))+\sqrt{D_u(r)}}(1-e^{-\lambda\tilde{t}})-\lambda R(\tilde{t})\right)\left(1+V_s'(\overline{u}(\tilde{t},D_u(r)))\frac{e^{-\lambda\tilde{t}}}{1-e^{-\lambda\tilde{t}}}\right)\\
&-\left(t-R(t)\right)\left(\frac{2}{a(\tilde{t},D_u(r))+\sqrt{D_u(r)}}+\frac{\lambda}{1-e^{-\lambda\tilde{t}}}\right) \geq 0\\
\implies &\frac{2(a(\tilde{t},D_u(r))-R(\tilde{t}))}{a(\tilde{t},D_u(r))+\sqrt{D_u(r)}}(1-e^{-\lambda\tilde{t}})-\lambda R(\tilde{t}) \geq 0\\
\iff &\frac{2(a(\tilde{t},D_u(r))-R(\tilde{t}))}{a(\tilde{t},D_u(r))+\sqrt{D_u(r)}} \geq 1-R'(\tilde{t}) && (45)
\end{aligned}
$$

The second equivalence is obtained by multiplying both sides by $a(\tilde{t},D_u(r))$ and dividing both sides by $\rho R(\tilde{t})$, and then using the following two identities: (i) from r's IC constraint that

$$\left(a(\tilde{t},D_u(r))-\sqrt{D_u(r)}\right)\left(a(\tilde{t},D_u(r))+\sqrt{D_u(r)}\right)=\rho R(\tilde{t}),$$

and (ii) $\frac{1-R'(\tilde{t})}{R(\tilde{t})}=\frac{\lambda}{1-e^{-\lambda\tilde{t}}}$. The implication follows from $t>R(t)$ and $V_s'(\cdot)>0$ from Claim 10. The last equivalence again uses the identity $\frac{1-R'(\tilde{t})}{R(\tilde{t})}=\frac{\lambda}{1-e^{-\lambda\tilde{t}}}$.

Now suppose toward a contradiction that $\tilde{t}\notin L_{D_u(r),\lambda}$, i.e. $a(\tilde{t},D_u(r))=\sqrt{D_u(r)+\rho R(\tilde{t})}<R(\tilde{t})+\rho/4$. Notice that the LHS of (45) is strictly increasing in $a(\tilde{t},D_u(r))$. Thus substituting $R(\tilde{t})+\rho/4$ for $a(\tilde{t},D_u(r))$ in (45) gives,

$$
\begin{aligned}
&\frac{\rho/2}{R(\tilde{t})+\rho/4+\sqrt{D_u(r)}} > 1-R'(\tilde{t})\\
\implies &\frac{\rho/2}{R(\tilde{t})+\rho/2} > 1-R'(\tilde{t}). && (46)\\
& && (47)
\end{aligned}
$$

The implication follows from $D_u(r) > \rho^2/16$ which in turn follows from Lemma 5 and $u \geq \rho^2/16$. Notice that $L_{\rho^2/16,\lambda} \subset L_{u,\lambda}$ $\forall u \geq \rho^2/16$. So because $D_u(r) > u^2/16$, proving a contradiction of $\tilde{t} \notin L_{D_u(r),\lambda}$ and (46)—which is independent of u—is implied by proving a contradiction between $\tilde{t} \notin L_{\rho^2/16,\lambda}$ and (46). Because the LHS of (46) is decreasing in $\tilde{t}$ and the RHS is increasing in $\tilde{t}$, it is sufficient to prove a contradiction for the case where $\tilde{t} = \bar{t}_{\rho^2/16,\lambda} = \max L_{\rho^2/16,\lambda}$. By the definition of $L_{\rho^2/16,\lambda}$ this means that

$$\begin{aligned}
\sqrt{\rho^2/16 + \rho R(\tilde{t})} &= \rho/4 + R(\tilde{t}) \\
\iff R(\tilde{t}) &= \rho/2 \\
\iff \frac{\lambda\tilde{t}}{e^{\lambda\tilde{t}} - 1} &= \frac{2 - \rho\lambda}{2}
\end{aligned} \tag{48}$$

Plugging this into (46) and expanding the expression for $R'(\tilde{t})$ gives

$$\begin{aligned}
&\frac{\rho/2}{R(\tilde{t}) + \rho/2} > 1 - \frac{\lambda(\tilde{t} - R(\tilde{t}))}{e^{\lambda\tilde{t}} - 1} \\
\iff &\frac{\lambda(\tilde{t} - \rho/2)}{e^{\lambda\tilde{t}} - 1} > 1/2 \\
\iff &\frac{2 - \rho\lambda}{2} - \frac{2\rho - \rho^2\lambda}{4\tilde{t}} > 1/2 \\
\iff &2\tilde{t}(1 - \rho\lambda) - 2\rho + \rho^2\lambda > 0.
\end{aligned}$$

Thus, since $\lambda < 2/\rho$ in this case, we have a contradiction, i.e., the claim is proved, if $\lambda\rho \geq 1$. Thus assume $\lambda\rho < 1$. Using the second order Taylor expansion of e^x gives that $e^{\lambda\tilde{t}} - 1 > \lambda\tilde{t} + \frac{\lambda^2\tilde{t}^2}{2}$. Using this in (48) gives, $\tilde{t} < \frac{2\rho}{2-\lambda\rho}$. Using this inequality in the last line of the display gives,

$$\begin{aligned}
2\frac{2\rho}{2 - \lambda\rho}(1 - \lambda\rho) &> \rho(2 - \lambda\rho) \\
4 - 4\lambda\rho &> (2 - \lambda\rho)^2 \\
0 &> 2\lambda\rho + \lambda\rho^2,
\end{aligned}$$

which is a contradiction and proves the claim.

Note that by Lemma 5 and because $u \geq \rho^2/16$, $D_u(r) \geq \rho^2/16$ $\forall r \in [0, t]$. This means that $L_{D_u(r),\lambda}$ remains of the form $[0, \bar{t}_{D_u(r),\lambda})$ per Lemma 15. First note that $t \in \{r \in [0, r] : t - r \in L_{D_u(r),\lambda}\}$ and so this set is non-empty. Let $r^* \in [0, t]$ be the minimum type r with $t - r \in L_{D_u(r),\lambda}$.[37] That is, $\forall r < r^*$, $t - r \notin L_{D_u(r),\lambda}$. By Claim 16, $\frac{d\,L^P(x_r)}{dr} \leq 0$ and so

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[37] Such a minimum exists because if $t - r_n \in L_{D_u(r),\lambda}$ for each $r_n \to r$ then $\sqrt{D_u(r_n) + \rho R(t - r_n)} \geq$

$L^P(x_{r^*}) \leq \tilde{V}(t,u)$. Now note that since the continuing allocations are the same on $[0,r^*]$, $L^P(x_{r^*}) - V_s(u) = e^{-\lambda r^*}(\tilde{V}(t-r^*, D_u(r^*)) - V_s(D_u(r^*)))$. Because of parts 1 and 2, and because $t - r^* \in L_{D_u(r^*),\lambda}$, it holds that $\tilde{V}(t-r^*, D_u(r^*)) \geq V_s(D_u(r^*))$. This means that $\tilde{V}(t,u) \geq V_s(u)$, completing the proof of part 3 and thereby the theorem. *Q.E.D.*

D.4. Proof of Lemma 6

Proof. Note that the pooling continuing allocation is not optimal at $u > \rho^2/16$ by Theorem 2. Assume the pooling allocation is optimal at some $u \leq \rho^2/16$. It must be better than an alternative continuing allocation which introduces a small separating portion at the beginning of the allocation. That is, adapting the condition of Lemma 14 using $\bar{t} = \infty$, $\underline{t} = 0$, and $\tilde{x} = \sqrt{u + \rho/\lambda}$ gives,

$$\begin{aligned}
& 1 - \frac{2(\sqrt{u+\rho/\lambda} - 1/\lambda)}{\sqrt{u+\rho/\lambda} + \sqrt{u}} \leq 0 \\
\iff & \sqrt{u+\rho/\lambda} - \sqrt{u} \geq 2/\lambda \\
\Longleftarrow & \rho/\lambda \geq 4/\lambda^2 \\
\iff & \lambda\rho \geq 4,
\end{aligned}$$

Q.E.D.

D.5. Proof of Corollary 1

Proof.

Claim 17. *Take an arbitrary incentive compatible allocation* x*. If* $\exists \tilde{t} : \underline{\tau}(\tilde{t}) > \rho/8$*, then there exists an allocation* y *that improves on* x *with the property that* y *is separating after* $\underline{\tau}(\tilde{t})$*, i.e.* $y(t') = d_{L^A(\tilde{t}|y)}(t) + \tilde{t} \; \forall t \geq \underline{\tau}(\tilde{t})$*.*

Proof of Claim: Suppose that for some $t < \tilde{t}$, $x(t) - t > \rho/4$, then clearly for the left adjacent threshold, $x(\underline{\tau}(t)) - \underline{\tau}(t) > \rho/4$. This means that $L^A(\underline{\tau}(t)|x) > \rho^2/16$, which would imply that separating is optimal following this threshold by Theorem 2 and proves the claim. Now suppose that $x(t) - t \leq \rho/4 \; \forall t \leq \tilde{t}$. But this means that $L^A_t(t|x) = \rho - 2(x(t) - t) \geq \rho/2 \; \forall t < \tilde{t}$. Thus, because $\underline{\tau}(\bar{t}) > \rho/8$, then $L^A(\underline{\tau}(\tilde{t})) > \rho^2/16$ and one can replace x with a separating continuing allocation above $\underline{\tau}(\tilde{t})$, and improve on x by Theorem 2. This proves the claim.

Take a sequence of incentive compatible allocations x_n such that $L^P(x_n) \to L$ where L is the infimum loss of the principal. Take $t_n \equiv \min\{\underline{\tau}_{x_n}(t) : \; t \in T, \; L^A(\underline{\tau}_{x_n}(t)|x_n) \geq \rho^2/16\}$,

---

$\rho/4 + R(t - r_n)$ for each r_n by definition of $L_{D_u(r),\lambda}$. Since both sides of this inequality are continuous in r_n, the inequality also holds for r.

where $t_n = \infty$ if the relevant set is empty. Without loss of optimality by Theorem 2, replace $\tilde{x}_n(t)$ with

$$y_n(t) \equiv \begin{cases} x_n(t) & t < t_n \\ d_{L^A(t_n|x_n)}(t) + t_n & t \geq t_n \end{cases}.$$

Suppose first there exists a subsequence t_n that converges to some $\bar{t}$. Then the optimal allocation is found by optimizing the allocation on $[0, \bar{t}]$ assuming the separating continuation loss above $\bar{t}$, which exists by Lemma 3 and the fact that the separating continuation loss is continuous in the allocation on $[0, \bar{t}]$. In the other case $t_n \to \infty$, and eventually for n greater than some n, $s_n \equiv \max\{\underline{\tau}_{y_{n_k}}([0, t_{n_k})\} \leq \rho/8$, otherwise by the logic of the claim above $L^A(s_n|y_n) > \rho^2/16$, violating the definition of t_n. Take a convergent subsequence of s_n that converges to some $\bar{s}$. Thus, an optimal allocation is found by optimizing the allocation on $[0, \bar{t}]$ assuming the pooling continuation value above $\bar{t}$. These are exhaustive cases and so an optimum exists.

Now suppose $\rho\lambda \leq 4$. By the claim above, if the optimal allocation has a threshold above $\rho/8$ then it is eventually separating. If it does not, then it is eventually pooling which is suboptimal by Lemma 6.

Q.E.D.

D.6. Proof of Proposition 3

Proof. Note that the inequality in Lemma 14 implies that for x^* optimal, it must be that $x^*(t) > r^*_{x^*}(t)$. Expanding this condition gives $\sqrt{L^A(\underline{\tau}(t)|x^*) + \rho R(\overline{\tau}(t) - \underline{\tau}(t))} > R(\overline{\tau}(t) - \underline{\tau}(t))$. Since the reputation is decreasing in λ, this means that $\forall\lambda$ small, any pooling interval in the continuing allocation at any u has bounded length. Moreover by inspecting this condition, this bound is uniform for small initial losses $L^A(\underline{\tau}(t)|x^*)$. Since the optimal continuing allocation is separating at initial loss $u > \rho^2/16$, this will imply that there exists some M, and $\overline{\lambda} > 0$ such that the optimal continuing allocation $\forall\lambda \leq \overline{\lambda}$ solves,

$$\min_{x \in IC([0,M]:\ L^A(0|x)=u)} \int_0^M (x(t') - t')^2 \lambda e^{-\lambda t'} dt' + e^{-\lambda M} V_s(L^A(M|x)).$$

I complete the proof in three steps. First I extend the result in Theorem 2 to $u \geq \rho^2/36$ in the uniform limit. Second I bound the derivative of the minimized continuation loss in the uniform limit. Third, I show that under this conclusion the solution to (4) always chooses the loss at the first threshold to be greater than $\rho^2/36$.

Step 1: $\forall\varepsilon > 0\ \exists\overline{\lambda} > 0$ such that $\forall\lambda \leq \overline{\lambda}$, $V(u) = V_s(u)\ \forall u \geq \rho^2/36 + \varepsilon$.

Recall that $\tilde{V}(t,u)$ is the principal's loss from a continuing allocation at u which pools $[0,t]$ and separates thereafter.

Claim 18. *There exists $\overline{\lambda}_0 > 0$, there exists $\overline{t} > 0$ such that $\forall u \leq \rho^2/4$, $\lambda < \overline{\lambda}$, $t > \overline{t}$, $\tilde{V}(t,u) > V_s(u)$.*

Proof of Claim: Notice that because $R(t) < t/2$ the agent loss at the first threshold $\overline{u}(a,t) > (\sqrt{u+\rho t/2} - t)^2 + \rho t/2$ across λ and diverges for large t. Take $\overline{t}_0$ large that this lower bound is greater that $\rho^2/4$. This means that $V_s(\overline{u}(t,u)) > \rho^2/4$; moreover, because $u < \rho^2/4$ $D_u(t) < \rho^2/4$ $\forall t$ and $V_s(D_u(t)) < \rho^2/4$— both points by Lemma 2. This means that the loss from separating at u after $t > \overline{t}_0$ is lower that for the initial pooling allocation. Thus we only need to prove that the separating allocation does better on the initial interval $[0,t]$. Notice that the pooling loss is higher than if the action for this initial interval were the expectation, i.e., respectively $R(t)$ and

$$\frac{1}{1-e^{-\lambda t}} \int_0^t (R(t) - \tilde{t})\lambda e^{-\lambda \tilde{t}} d\tilde{t}. \tag{49}$$

Again because of Lemma 2 $D_u(t) < \rho^2/4$ and so the separating loss on this initial interval is also bounded above by $\rho^2/4$. But note that the loss in (49) is the variance on this initial interval which (i) is decreasing in λ, and (ii) grows in t, and (iii) grows unboundedly as $t \to \infty$ and $\lambda \to 0$. Thus by taking $\overline{t}$ large enough and $\overline{\lambda}_0$ small enough so that (49) is greater than $\rho^2/4$ proves the claim.

Because of Claim 18, I only need to show that $\tilde{V}(t,u) > V_s(u)$ on some initial compact interval of types $[0,\overline{t}]$. I will complete this step by showing that the derivative of $\tilde{V}(t,u)$ in t is strictly positive, i.e., the inequality in (32), in this region.

Claim 19. *For every $\varepsilon > 0$, there exists $\overline{\lambda}_1$ such that $\forall t \in [0,\overline{t}]$, $\lambda < \overline{\lambda}_1$, and $u \in [\rho^2/36+\varepsilon, \rho^2/16]$ such that (32) holds.*

Proof of Claim:

Let $\overline{U} \equiv \max_{t\in[0,M],u\in[\rho^2/36+\varepsilon,\rho^2/16],\lambda\in(0,\overline{\lambda}]} \overline{u}(t,u)$ so that $\overline{u}(t,u)$ lies in $[\rho^2/36, \overline{U}]$. Note that $\lim_{\lambda\to 0} V_s(\tilde{u}) = \rho^2/4$ for any $\tilde{u}$ because as $\lambda \to 0$, the distribution places higher weight on higher types and $D_u(t) \to \rho^2/4$ as $t \to \infty$ by Lemma 2. Moreover since $V_s(\tilde{u})$ is increasing in $\tilde{u}$, this convergence is uniform for $\tilde{u} \in [\rho^2/36, \overline{U}]$. Let $V_0'(u) \equiv \lim_{\lambda\to 0} V'(u)/\lambda$. Note that by Claim 10 and Theorem 2, for $\tilde{u} > \rho^2/16$,

$$V_0'(\tilde{u}) = \lim_{\lambda\to 0} V_s'(\tilde{u})/\lambda = \lim_{\lambda\to 0} \frac{V_s(\tilde{u}) - \tilde{u}}{\rho - 2\sqrt{\tilde{u}}} = \frac{\rho + 2\sqrt{\tilde{u}}}{4}.$$

Similarly because $V_s'(\tilde{u})$ is increasing by Claim 11 and the above limit is continuous in $\tilde{u}$, $V_s(\tilde{u})$ converges uniformly to the above limit for $\tilde{u} \in [\rho^2/36, \overline{U}]$.. Now recall that if (32) holds for all $t \in [0, M]$ and $u \in [\rho^2/36 + \varepsilon, \rho^2/16]$ then separating is optimal. Take the limit of the LHS of (32) as $\lambda \to 0$. Let $\underline{a} \equiv \lim_{\lambda \to 0} a(t, u) = \sqrt{u + \rho t/2}$, $\underline{d}_0 \equiv \lim_{\lambda \to 0} d_0 = \underline{a} - t$, and $\underline{d}_1 \equiv \lim_{\lambda \to 0} d_1 = \sqrt{\underline{d}_0^2 + \rho t/2}$. The limit of the LHS of (32) is given by

$$\lim_{\lambda \to 0} \left[\frac{V_s(d_1^2) - d_1^2}{\rho - 2d_1} \left(\frac{\frac{t}{R(t)}(2(d_0 + d_1) - \rho) + \rho}{(d_0 + d_1)^2} + \lambda \right) \right]$$
$$= \frac{\rho + 2\underline{d}_1}{4} \frac{4(\underline{d}_0 + \underline{d}_1) - \rho}{(\underline{d}_0 + \underline{d}_1)^2}.$$

The reason this convergence is uniform across $t \in [0, M]$ and $u \in [\rho^2/36 + \varepsilon, \rho^2/16]$ is as follows. First, as stated above, $V_s'(u)/\lambda$ converges uniformly to $\frac{\rho + 2\underline{d}_1}{4}$. Second, note that for $k = \lambda t$,

$$t/R(t) = \frac{k(e^k - 1)}{e^k - 1 - k},$$

which converges to 2 as $k \to 0$. Thus, $k < \overline{\lambda}_1 \bar{t}$ for all $t \leq \bar{t},\ \lambda < \overline{\lambda}$, and $t/R(t) \to 2$ uniformly across t. Next note that d_0 and d_1 are Lipschitz continuous functions of (t, u) on this region with a constant independent of $\lambda \in (0, \overline{\lambda}_1]$. Moreover, $d_1 + d_0$ is uniformly bounded away from 0 across λ and $u \geq \rho^2/36$, and $t \geq 0$, so that the functions $\frac{2(d_1 + d_0) - \rho)}{(d_1 + d_0)^2}$, $\frac{\rho}{(d_0 + d_1)^2}$, and $V_s'(d_1)/\lambda$ are all Lipschitz continuous with constant independent of $\lambda \in (0, \overline{\lambda}]$, and so we can apply the Arzela-Ascoli Theorem to obtain uniform convergence.[38]

The above expression is greater than 1 if

$$\begin{aligned}
& 2(\underline{d}_0 + \underline{d}_1)\underline{d}_1 + \rho(\underline{d}_0 + \underline{d}_1) - \rho/2\underline{d}_1 - \rho^2/4 > (\underline{d}_0 + \underline{d}_1)^2 \\
\iff & \underline{d}_1^2 + \rho/2\underline{d}_1 + \rho\underline{d}_0 - \rho^2/4 > \underline{d}_0^2 \\
\iff & \underline{d}_1^2 + \rho/2\underline{d}_1 + \rho\underline{d}_0 - \rho^2/4 > \underline{d}_1^2 - \rho t/2 \\
\iff & 2t + 2\underline{d}_1 + 4\underline{d}_0 - \rho > 0
\end{aligned}$$

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[38] To expand on these points recall $d_0 = \sqrt{u + \rho R(t)} - t$ and $d_1 = \sqrt{d_0^2 + \rho(t - R(t))}$. Thus $\frac{d(d_0)}{dt} = \frac{\rho R'(t)}{2\sqrt{u + \rho R(t)}} < \frac{3}{2}$ because $R'(t) < 1/2$, and $u > \rho^2/36$, and $\frac{d(d_0)}{du} = \frac{1}{2\sqrt{u + \rho R(t)}} < 3$. Moreover, $\frac{d(d_1)}{du} = \frac{d_0 \frac{d(d_0)}{du}}{d_1}$ and $\frac{d(d_1)}{dt} = \frac{\rho R'(t) d_0 \frac{d(d_0)}{du} + \rho(1 - R'(t))/2}{d_1}$ which is uniformly bounded as d_0 is continuous and the domain is compact, $d_1 \geq u > \rho^2/36$ by Lemma 5, and $0 < R'(t) < 1/2$. Finally, $d_1 + d_0$ is uniformly bounded away from 0 because (i) if $d_0 > 0$ then $d_1 + d_0 > d_1 \geq u > \rho^2/36$ by Lemma 5, and (ii) if d_0 is negative then note that $t > \sqrt{u} > 1/6$, and by the concavity of the square root function $d_1 + d_0 > \frac{\rho(t - R(t))}{2d_1} > 0$, which then gives the uniform bound by the compactness of the domain.

Note first that the last line is strictly increasing in u—its derivative is given by $\frac{\underline{d}_0+2\underline{d}_1}{\underline{d}_1\sqrt{u+\rho t/2}}$, and $\underline{d}_1 > 0$ and $\underline{d}_1 + \underline{d}_1 \geq \rho^2/4$ by Lemma 15. Thus it is sufficient to prove the inequality is weak taking $u = \rho^2/36$, i.e., expanding the definitions of $\underline{d}_0$ and $\underline{d}_1$,

$$\begin{aligned} & 2t + 2\underline{d}_1 + 4\underline{d}_0 - \rho \geq 0 \\ \Longleftarrow\ & \underline{d}_1^2 \geq (\rho/2 - 2\underline{d}_0 - t)^2 \\ \Longleftrightarrow\ & \rho^2/3 + 2\rho t - 2\underline{a}t - 2\rho\underline{a} \leq 0 \end{aligned}$$

Note that the LHS of the last line is decreasing in t—its derivative is given by

$$2\rho - 2\underline{a} - \frac{\rho^2}{2a} - \frac{\rho t}{2\underline{a}} = -\frac{1}{2a}\left((2a-\rho)^2 + \rho t\right) < 0.$$

Thus, proving the inequality at $t = 0$ is sufficient. Evaluating the LHS at $t = 0$ gives $\rho^2/3 - 2\rho\sqrt{\rho^2/36} = 0$, completing step 1.

Step 2: I will show that $\forall \varepsilon > 0$, $\exists \overline{\lambda} > 0$ such that $\forall u \leq \rho^2/16$, $\forall \lambda < \overline{\lambda}$, $V'(u)/\lambda \leq 3\rho/8 + \varepsilon$.

Consider the alternative representation of allocations by their interval partition introduced in Lemma 13, i.e. $x \in IC([0, M])$ is identified with its threshold function $\tau : [0, M] \to [0, M]$. Recall the set of all allowable threshold functions to be $\mathcal{R}$ and that $x_{\tau,u}$ is the unique continuing allocation at u with threshold function $\tau \in \mathcal{R}$ by Lemma 13. With this one can write

$$V(u) = \min_{\tau\in\mathcal{R}} \int_0^M (x_{\tau,u}(t') - t')^2 \lambda e^{-\lambda t'} dt' + e^{-\lambda M} V_s(L^A(M|x_{\tau,u})) \tag{50}$$

Thus, by Lemma 13 and the envelope theorem, $V'(u) = \frac{d\,L^P(x_{\tau,u})}{du}$ for some optimal $\tau \in \mathcal{R}$ in (50).

Suppose step 2 does not hold. This means there exists $\varepsilon > 0$, and sequences u_n, λ_n, and $\tau_n \in \mathcal{R}$ such that $\forall n$, τ_n is optimal in (50) under u_n and λ_n, $u_n \leq \rho^2/16$, $\lambda_n \to 0$, and $\frac{d\,L^P(x_{\tau_n,u_n})}{du}/\lambda_n \geq 3\rho/8 + \varepsilon$.

Let u_n without loss be the maximum such loss to satisfy the above conditions under λ_n.[39] First consider the case in which there is an infinite subsequence such that a first threshold $t_n > 0$ exists under τ_n. Since $u_n \in [0, \rho^2/16]$ and $t_n \in [0, M]$ are both in compact sets there exists a further subsequence such that $u_n \to \tilde{u}$, and $t_n \to \tilde{t}$. Note that $\tilde{u} \leq \rho^2/36$ from step 1. Otherwise $V(u_n)$ is separating for large enough n, and $V'(u_n) = V_0'(u_n)$ satisfies the bound. Using the definition of $a(t_n, u_n)$ and $\overline{u}(t_n, u_n)$, one can compute,

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[39] $\frac{d\,L^P(x_{\tau,u})}{du}$ is continuous in u by the logic of Lemma 13, so this is well defined.

$$\begin{aligned}&\frac{dL^P(x_{\tau_n,u_n})}{du}/\lambda_n\\ =&\frac{(a(t_n,u_n)-R(t_n))(1-e^{-\lambda_n t_n})}{\lambda_n a(t_n,u_n)}+\frac{(a(t_n,u_n)-t_n)}{a(t_n,u_n)}e^{-\lambda t_n}V'(\overline{u}(t_n,u_n))/\lambda_n > 3\rho/8+\varepsilon\end{aligned} \tag{51}$$

Suppose first that $\tilde{t}=0$. This, combined with $\tilde{u}\le \rho^2/36$, means that $u_n<\overline{u}(t_n,u_n)<\rho^2/16$. By the fact that u_n is the maximal violation under λ_n, this means that $V'(\overline{u}(t_n,u_n))/\lambda_n < 3\rho/8+\varepsilon$. Using this in (51) gives

$$(a(t_n,u_n)-R(t_n))-(3\rho/8+\varepsilon)\frac{\lambda_n t_n}{e^{\lambda_n t_n}-1} > \lambda_n a(t_n,u_n)(3\rho/8+\varepsilon). \tag{52}$$

This first term on the LHS goes to $\sqrt{\tilde{u}}$ as $\lambda_n>0$ and $t_n\to 0$. However, the second term on the LHS goes to $-3\rho/8-\varepsilon$ as $\lambda_n t_n\to 0$. Since the RHS is positive and $\sqrt{\tilde{u}}<3\rho/8$, this is a contradiction.

Because it is a similar argument, consider now the case in which for some small λ_n there is no first threshold under τ_n. That means there is a sequence of thresholds s_k under τ_n, such that $s_k\to 0$. Take $k>K$ large enough so that $L^A(s_k|x_{\tau_n,u_n})<\rho^2/16$. This is guaranteed by the approximation in Claim 9. Since u_n was the maximal violation under λ_n, it must be that $V'(L^A(s_k|x_{\tau_n,u_n}))/\lambda_n<(3\rho/8+\varepsilon)\ \forall k>K$. Let $\tilde{y}\equiv\sqrt{u_n+s_k(\rho-2\sqrt{u_n})+s_k^2}+s_k$. Thus,

$$\begin{aligned}&\frac{dL^P(x_{\tau_n,u_n})}{du}/\lambda_n > 3\rho/8+\varepsilon\\ \iff&\frac{1}{\lambda_n}\int_0^{s_k}\frac{d\ L^A(t'|x_{\tau_n,u_n})}{du}\lambda_n e^{-\lambda_n t'}+e^{-\lambda_n s_k}\frac{d\ L^A(s_k|x_{\tau_n,u_n})}{du}V'(L^A(s_k|x_{\tau_n,u_n}))/\lambda_n > 3\rho/8+\varepsilon\\ \implies&\frac{(\tilde{y}-R(s_k))(1-e^{-\lambda_n s_k})}{\lambda_n\tilde{y}}+e^{-\lambda_n s_k}\frac{\tilde{y}-s_k}{\tilde{y}}V'(L^A(s_k|x_{\tau_n,u_n})) > 3\rho/8+\varepsilon\\ \implies&\tilde{y}-R(s_k)-\frac{\lambda_n s_k}{e^{\lambda_n s_k}-1}(3\rho/8+\varepsilon) > \lambda_n\tilde{y}(3\rho/8+\varepsilon).\end{aligned}$$

First note that, given the implications above hold, then the last line is a contradiction for the same reasons as for (52). To see the first implication recall the recursion from Lemma 13 that

$$\begin{aligned}&\frac{d\ L^A(t|x_{\tau_n,u_n})}{du}=\\ &\frac{x_{\tau_n,u}(t)-t}{x_{\tau_n,u_n}(t)-\underline{\tau}_n(t)}\left(\prod_{(\underline{t},\overline{t})\in\tilde{J}^p:\overline{t}<t}\frac{x_{\tau_n,u_n}(\underline{t})-\overline{t}}{x_{\tau_n,u_n}(\underline{t})-\underline{t}}\right)\left(\prod_{(\underline{t},\overline{t})\in\tilde{J}^s:\underline{t}<t}e^{2/\rho\left((x_{\tau_n,u_n}(\underline{t})-\underline{t})-(x_{\tau_n,u_n}(\max\{\overline{t},t\})-\max\{\overline{t},t\})\right)}\right).\end{aligned}$$

Note that each term in the product is positive for $t' < s_k < \overline{\delta}$ because $x(t') > \overline{\tau}_n(t')$ by definition of $\overline{\delta}$. Thus each term in the product is less than 1 and thereby increasing in $x_{\tau_n,u_n}(t)$ $\forall t$. Thus if $\tilde{y} > x_{\tau_n,u_n}(t) \forall t$, replacing every $x_{\tau_n,u_n}(t)$ by $\tilde{y}$ delivers that

$$\frac{d\ L^A(t|x_{\tau_n,u_n})}{du} < \frac{\tilde{y}-t}{\tilde{y}}.$$

To see why $\tilde{y} > x_{\tau_n,u_n}(t)$, note that by incentive compatibility,

$$\begin{aligned} &(x_{\tau_n,u_n}(s_k^-) - s_k)^2 + \rho(s_k - r^*(s_k^-)) < (x_{\tau_n,u_n}(0) - s_k)^2 + \rho(s_k - r^*(0)) \\ \Longrightarrow\ &(x_{\tau_n,u_n}(s_k^-) - s_k)^2 < (\sqrt{u_n + \rho r^*(0)} - s_k)^2 + \rho(s_k - r^*(0)) \\ \Longrightarrow\ &x_{\tau_n,u_n}(s_k) < \sqrt{\sqrt{u_n} - \overline{\tau}_x(t))^2 + \rho s_k} + s_k = \tilde{y}. \end{aligned}$$

The first implication is due to $s_k > r^*(s_k^-)$ because s_k is a threshold of x_{τ_n,u_n}. the second implication is due to the fact that the RHS is decreasing in $r^*(0)$.

Now consider the case in which there is an infinite subsequence of first thresholds $t_n \to \tilde{t} > 0$. First suppose that $\overline{u}(\tilde{t},\tilde{u}) > \rho^2/36$, so that the allocation above the first threshold is eventually separating. Then as $n \to \infty$, the inequality in (51) converges to

$$\frac{(\sqrt{\tilde{u} + \rho\tilde{t}/2} - \tilde{t}/2)\tilde{t}}{\sqrt{u + \rho\tilde{t}/2}} + \frac{\sqrt{\tilde{u} + \rho\tilde{t}/2} - \tilde{t}}{\sqrt{\tilde{u} + \rho\tilde{t}/2}} \left(\frac{\rho + 2\sqrt{\overline{u}(\tilde{t},\tilde{u})}}{4} \right) > 3\rho/8 + \varepsilon. \tag{53}$$

Notice that the LHS is increasing in $\tilde{u}$ for any t and so it is without loss to evaluate the LHS at $\tilde{u} = \rho^2/36$, which can be shown to be strictly less the $3\rho/8$ for all $t \geq 0$.

Now consider that $\overline{u}(\tilde{t},\tilde{u}) \leq \rho^2/36$ so that $V_0'(\overline{u}(\tilde{t},\tilde{u})) \leq 3\rho/8 + \varepsilon$ by the assumption that u_n was taken as the maximal violator. Consider first the case in which $a(\tilde{t},\tilde{u}) - \tilde{t} \geq 0$. But then,

$$\frac{(\sqrt{\tilde{u} + \rho\tilde{t}/2} - \tilde{t}/2)\tilde{t}}{\sqrt{\tilde{u} + \rho\tilde{t}/2}} + \frac{\sqrt{\tilde{u} + \rho\tilde{t}/2} - \tilde{t}}{\sqrt{\tilde{u} + \rho\tilde{t}/2}} 3\rho/8 \geq 3\rho/8$$

Notice again that the LHS is increasing in $\tilde{u}$ and so it is without loss to evaluate the inequality at $\tilde{u} = \rho^2/36$ and this leads to a contradiction for any $t \geq 0$. Lastly, suppose $a(\tilde{t},\tilde{u}) - \tilde{t} < 0$ which means $\sqrt{\tilde{u} + \rho\tilde{t}/2} < \tilde{t} \implies \tilde{t} > \rho/2$. But then $\overline{u}(\tilde{t},\tilde{u}) > \rho^2/4$ which is a contradiction. This completes step 2.

Step 3: I will show that there exists $\overline{\lambda} > 0$, and small enough $\varepsilon > 0$, such that the optimal first threshold and first action t, a in (4), induce first thresholds loss in (4) $u \equiv (a-t)^2 + \rho(t - R(t)) \geq \rho^2/36 + \varepsilon$. Step 1 then provides that the optimal allocation is separating after the first threshold. Take small $\varepsilon > 0$ and $\overline{\lambda}$ small enough to satisfy step 1 and step 2, with $\lambda < \overline{\lambda}$. At an optimum first threshold t and first action $a \leq t$.[40] in (4) the loss is increasing in t_1 given $u_1 \leq \rho^2/36$.

$$\big((a_1 - t_1)^2 - V(u_1)\big) + \left(\rho(1 - R'(t_1)) + 2(t_1 - a_1)\right) \frac{V'(u_1)}{\lambda} \geq 0 \tag{54}$$

$$\implies \big((a_1 - t_1)^2 - V(u_1)\big) + \left(\rho(1 - R'(t_1)) + 2(t_1 - a_1)\right)(3\rho/8) \geq 0. \tag{55}$$

The first line is the derivative of (4) in t. The implication follows directly from the conclusion of step 2. Towards a contradiction, suppose that there exists a sequence λ_n, first action and threshold a_n, t_n, with induced first threshold loss u_n, such that $\lambda_n \to 0$, $u_n \leq \rho^2/36 \forall n$, and (55) $\forall n$. Notice that since $u_n \geq \rho(t_n - R(t_n)) > \rho t_n/2$, the t_n sequence is bounded above, and so because $a_n \leq t_n$, the a_n sequence is also bounded above. Thus because both t_n and a_n lie in compact intervals they admit convergent subsequences, for which (abusing notation) we refer to as the original sequence. Thus $t_n \to \tilde{t}$ and $a_n \to \tilde{a} \leq \tilde{t}$, and since, as shown in step 1, $R(t)$ converges to $t/2$ uniformly over any compact interval of t as $\lambda \to 0$, $u_n \to \tilde{u} = (\tilde{a} - \tilde{t})^2 + \rho t/2 \leq \rho^2/36$. Since $V(u)$ is increasing in u and converges to $\rho^2/4$ for any u as discussed in step 1, $V(u)$ converges uniformly to $\rho^2/4$ for u in any compact interval as $\lambda \to 0$, and so $V(u_n)$ converges to $V(\tilde{u}) = \rho^2/4$. Also note that $R'(t)$ is continuous and decreasing in t, converges to $1/2$ for any t as $\lambda \to 0$, and so $R'(t)$ converges uniformly for any compact interval of t as $\lambda \to 0$, i.e., $R'(t_n) \to 1/2$. Thus the fact that (55) holds along the sequence and its LHS is continuous in t_n, a_n implies that,

$$\begin{aligned} &\big((\tilde{a} - \tilde{t})^2 - \rho^2/4\big) + \big(\rho/2 + 2(\tilde{t} - \tilde{a})\big)(3\rho/8) \geq 0 \\ \iff &\frac{3\rho}{4}\sqrt{\tilde{u} - \frac{\rho}{2}\tilde{t}} \geq \frac{\rho^2}{16} + \frac{\rho}{2}\tilde{t} - \tilde{u}. \end{aligned}$$

The equivalence is obtained by using the identity $\tilde{u} = (\tilde{t} - \tilde{a})^2 + \rho\tilde{t}/2$ to substitute out for $\tilde{a}$. Now note that if the above inequality holds, it also holds for higher $\tilde{u}$. Note that since $\tilde{a} \leq \tilde{t}$, $\tilde{t}^2 + \rho\tilde{t}/2 \geq \tilde{u} \geq \rho\tilde{t}/2$. Using this upper bound, to convert the above inequality, and then using the lower bound in the last line gives,

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[40] Recall that if $a > t$, then decreasing a strictly decreases loss on the first interval, as the first interval optimal action is $R(t)$, and also decreases first threshold loss u, which then strictly decreases $V(u)$ by the alignment principle.

$$
\begin{aligned}
\frac{3\rho}{4}\tilde{t} &\geq \frac{\rho^2}{16} - \tilde{t}^2 \\
\implies \tilde{t} &\geq \frac{(\sqrt{13}-3)\rho}{8} \\
\implies \tilde{u} &\geq \frac{(\sqrt{13}-3)\rho^2}{16} > \frac{\rho^2}{36},
\end{aligned}
$$

but the last line is a contradiction because $\tilde{u} \leq \rho^2/36$. This completes the proof of step 3 and thereby the proposition. *Q.E.D.*

E. Proofs from Section 5

E.1. Proof of Proposition 4

Proof. Take a simple delegation set $\tilde{A} = A_1 \cup ... \cup A_n$ and two different incentive compatible allocations $x, y : T \to \tilde{A}$ such that $x(T) = \tilde{A}$. Note that x satisfies the D1 refinement trivially because there are no off-path actions. First I prove a claim about the implications of the D1 refinement in this context. It says that the off path belief for all unused intermediate actions under y must put probability 1 on a specific type defined as follows: $\forall a \in (\tilde{A} \setminus y(T)) \cap [0, y(M)]$, define $t(a) \equiv \inf\{t : \ y(t) > a\}$ which exists because by definition $a < \sup y(T)$.

Claim 20. *If y satisfies the D1 refinement, then $\forall t \in T$, $\forall a \in (\tilde{A} \setminus y(T)) \cap [0, y(M)]$,*

$$L^A(t|y) \leq (a-t)^2 + \rho(t - t(a)).$$

Proof of Claim: Take $a \in \tilde{A} \setminus y(T) \cap [0, y(M)]$. First I prove that for some $t' \in T$

$$\{\tilde{R} \in [0, M] : \ L^A(t'|x) > (a-t')^2 + \rho(t' - \tilde{R})\} \neq \emptyset, [0, M].$$

If this set is $[0, M]$, then there cannot be a reputation for a that makes $y(a)$ incentive compatible in the PBE sense. To see that the set is non-empty, note that If $a \in [0, M)$, then taking $t' = a$ and $\tilde{R} = M$ is a witness. If instead $a \geq M$, then taking $t' : \ y(t') > a$, which exists because $a \notin y(T)$, and $\tilde{R} = M$ is a witness.

Note that $\forall \tilde{R} \in [0, M]$ reputation, $L^A(t|y) - \Big((a-t)^2 + \rho(t - \tilde{R})\Big)$ is strictly increasing (decreasing) for $t < (>) t(a)$. This is because by Lemma 8, the right derivative

$$\begin{aligned}&\frac{d^+}{dt}\left(L^A(t|y) - \left((a-t)^2 + \rho(t - \tilde{R})\right)\right)\\ =&2(a - y^+(t)).\end{aligned}$$

This means that given any off-path reputation $\tilde{R}$ for a the loss difference between action a and any type's equilibrium loss is uniquely minimized at $t(a)$. Thus for every $t \neq t(a)$ the condition in (8) holds and so the off path belief must put 0 probability on all $t \neq t(a)$, i.e., the equilibrium reputation for a is $t(a)$ under, the D1 refinement, and so it must be that $L^A(t|y) \leq (a-t)^2 + \rho(t - t(a))\ \forall t \in T$.

Claim 21. *If $A_i \cap y(T)$ is not a singleton then $A_i \cap y(T) = [\min A_i, \tilde{a})$ or $A_i \cap y(T) = [\min A_i, \tilde{a}) \cup \{\overline{a}\}$ for some $\overline{a} > \tilde{a} > \min A_i$.*

Proof of Claim: Suppose A_i is not a singleton. I first prove that for $a < \max A_i \cap y(T)$, it cannot be that $(\underline{s}, \overline{s}) = y^{-1}(a)^{\mathrm{o}}$ with $\underline{s} < \overline{s}$. If this were the case then $r_y^*(\overline{s}) \geq \overline{s} > R(\underline{s}, \overline{s}) = r_y(a)$, and so $y(\overline{s}) \equiv a'' > a$. Note that $a'' \in y(T) \cap A_i$ because $a < \max A_i \cap y(T)$ and a'' is the lowest action in $y(T)$ greater than a. Because A_i is not a singleton, it is an interval, and so $(a, a'') \subset A_i$. This means that all $a' \in (a, a'')$ are unused under y, and so by Claim 20 bear reputations $\overline{s}$. Thus for small enough $\varepsilon > 0$, $a + \varepsilon$ gives approximate material loss and strictly lower reputation loss, and thereby strictly lower total loss than a for types in $(\underline{s}, \overline{s})$, contradicting D1 incentive compatibility of y. This means that for all actions lower than the maximum action, the allocation is separating, i.e. $A_i \cap y(T) = [\underline{a}, \tilde{a})$ or $A_i \cap y(T) = [\underline{a}, \tilde{a}) \cup \{\overline{a}\}$ for some $\overline{a} > \tilde{a} > \underline{a}$.

To see why $\underline{a} = \min A_i$, note that otherwise $\underline{a} - \varepsilon \in A_i \setminus y(T)$ is an unused action under y and by Claim 20 has reputation $y^{-1}(\underline{a})$ which is a singleton because y is separating on $(y^{-1}(\underline{a}), y^{-1}(\tilde{a}))$. Also because y is separating on this region, it holds that $\underline{a} > y^{-1}(\underline{a})$, and so for small $\varepsilon > 0$, $\underline{a} - \varepsilon$ is a profitable deviation for $y^{-1}(\underline{a})$ because it has lower material loss and the same reputation. This proves the claim.

Now take the type $t^* < M$ such that $x(t^*) \neq y(t^*)$, but $x(t') = y(t')\ \forall t' < t^*$. This is well defined because $\tilde{A}$ is a simple delegation set and because of Claim 21.

Let $\tilde{A}_i \equiv y(\{t' : t' > t^*\}) \cap A_j$ where A_j is the ith indexed interval such that $y(\{t' : t' > t^*\}) \cap A_j \neq \emptyset$. Let $\overline{t}_i^x \equiv \sup\ x^{-1}(\tilde{A}_i)$, $\underline{t}_i^x \equiv \inf x^{-1}(\tilde{A}_i)$, $\overline{t}_i^y \equiv \sup\ y^{-1}(\tilde{A}_i)$, and $\underline{t}_i^y \equiv \inf\ y^{-1}(\tilde{A}_i)$.

Claim 22. $\forall i\ |\tilde{A}_i| = 1$, *and either $\underline{t}_i^x \leq \underline{t}_i^y$ and $\overline{t}_i^x > \overline{t}_i^y$, or $\underline{t}_i^x \geq \underline{t}_i^y$ and $\overline{t}_i^x < \overline{t}_i^y$.*

Proof of Claim: I proceed by induction on i.

Base Step:

Let $\max\{y(t^*), x(t^*)\} \equiv a_1$ and $\min\{y(t^*), x(t^*)\} \equiv a_0$. First suppose that $a_0 = y(t^*) < x(t^*) = a_1$. Because $x(T) = \tilde{A}$, a_0 must be used under x. Let $\underline{s} = \inf x^{-1}(a_0) < t^*$. Since $x(t) = y(t)$ for $t < t^*$ by definition $\underline{s} = \inf y^{-1}(a_0)$ and therefore $y(t) = a_0$ if $t \in [\underline{s}, t^*)$ and $x(t) = a_0 \iff t \in [\underline{s}, t^*)$. Therefore since $y^{-1}(a_0)$ is a nondegenerate interval, by Claim 21 $a_0 = \max A_1 \cap y(T)$ and so since a_0 is also the minimum action in $\tilde{A}_1$ by definition, $\{a_0\} = \tilde{A}_1$ is a singleton. This means that $\underline{t}_1^x = \underline{t}_1^y = \underline{s}$, and because $y(t^*) = a_0 \neq x(t^*)$, $\overline{t}_1^x = t^*$ and $\overline{t}_1^y > t^*$.

Next suppose that $a_0 = x(t^*) < y(t^*) = a_1$. Let $\underline{s}_1^y$ $(\overline{s}_1^y) \equiv \inf\ (\sup) y^{-1}(a_1)$ and similarly $\underline{s}_0^x$ $(\overline{s}_0^x) \equiv \inf\ (\sup) x^{-1}(a_0)$. Because $a_0 = x(t^*)$, $\overline{s}_0^x \geq t^* \geq \underline{s}_0^x$. Since $x(t) = y(t)\ \forall t < t^*$, $y(t) \leq a_0\ \forall t < t^*$, so $\underline{s}_1^y = t^* \leq \overline{s}_0^x$. Since $x(T) = A$, a_1 is also used under x, so define $\underline{s}_1^x$ $(\overline{s}_1^x) \equiv \inf\ (\sup) x^{-1}(a_1)$. By monotonicity, $\overline{s}_0^x \leq \underline{s}_1^x$ and so $\underline{s}_1^x \geq \underline{s}_1^y$ In addition, because $x(t^*) = a_0$ and x is right continuous, this inequality is strict so $\underline{s}_1^x > \underline{s}_1^y$. I will show that $\overline{s}_1^x < \overline{s}_1^y$. Suppose that instead $\overline{s}_1^x \geq \overline{s}_1^y$, then by incentive compatibility,

$$\begin{aligned}
& L^A(a_0, \overline{s}_0^x | x) \leq L^A(a_1, \overline{s}_0^x | x) \\
\iff & (a_0 - \overline{s}_0^x)^2 - \rho R(\underline{s}_0^x, \overline{s}_0^x)) \leq (a_1 - \overline{s}_0^x)^2 - \rho R(\underline{s}_1^x, \overline{s}_1^x)) \\
\implies & (a_0 - \overline{s}_0^x)^2 - \rho R(\underline{s}_0^x, \overline{s}_0^x)) \leq (a_1 - \overline{s}_0^x)^2 - \rho R(\overline{s}_0^x, \overline{s}_1^y)) \\
\implies & (a_0 - \underline{s}_1^y)^2 - \rho R(\underline{s}_0^x, \underline{s}_1^y)) < (a_1 - \underline{s}_1^y)^2 - \rho R(\underline{s}_1^y, \overline{s}_1^y)) \\
\iff & L^A(a_0, \underline{s}_1^y | y) < L^A(a_1, \underline{s}_1^y | y),
\end{aligned}$$

The first implication is due to decreasing the reputation on the RHS as $\overline{s}_0^x \leq \underline{s}_1^x$ and by hypothesis $\overline{s}_1^x \geq \overline{s}_1^y$. The second implication invokes condition M^* moving the middle type $\overline{s}_0^x$ down to $\underline{s}_1^y = t^*$. Note that because $\underline{s}_1^x > \underline{s}_1^y$, either the change from the second to the third line, or from the third to the penultimate line results in the strict inequality. The RHS of the penultimate line is the equilibrium utility under y of $\underline{s}_1^y$ and is replaced as such on the last line. Thus the last line is a contradiction to incentive compatibility under y. The LHS of the penultimate line is equal to the LHS of the last line because the equilibrium reputation of a_0 under y is equal to the reputation on the LHS of the penultimate line, i.e., $r_y(a_0) = R(\underline{s}_0^x, \underline{s}_1^y))$. To see this, first consider that $\underline{s}_0^x < t^* = \underline{s}_1^y$. Then because $y(t) = x(t)\ \forall t < t^*$ and $y(t^*) = a_1$, $y^{-1}(a_0) = [\underline{s}_0^x, \underline{s}_1^y)$ and $r_y(a_0) = R(\underline{s}_0^x, \underline{s}_1^y)$. Suppose instead that $\underline{s}_0^x = \underline{s}_1^y = t^*$. Then a_0 is off-path under y, and $r_y(a_0) = \underline{s}_1^y = R(\underline{s}_0^x, \underline{s}_1^y))$ by Claim 20.

Thus it must be that $\overline{s}_1^x < \overline{s}_1^y$. This means that $y^{-1}(a_1)$ is a nondegenerate interval and so by Claim 21 it must be that $a_1 = \max A_i \cap y(T)$ and so since a_1 is also the minimum action in $\tilde{A}_1$ by definition, $\{a_1\} = \tilde{A}_1$ is a singleton. This means that $\underline{t}_1^x = \underline{s}_1^x > \underline{s}_1^y = \underline{t}_1^y = t^*$ and

$\overline{t}_1^x = \overline{s}_1^x < \overline{s}_1^y = \underline{t}_1^y = \underline{s}$ which proves the base case.

Inductive Step:

Suppose that $\tilde{A}_i = \{a_i\}$ is a singleton by the inductive hypothesis. Let $a_{i+1} \equiv y(\overline{t}_i^y)$. This means that a_{i+1} is the minimum action in $\tilde{A}_i$. Define $\underline{s}_{i+1}^y$ $(\overline{s}_{i+1}^y) = \inf y^{-1}(a_{i+1})$ $(\sup y^{-1}(a_{i+1}))$. Because $x(T) = \tilde{A}$, a_{i+1} is used under x as well and so one can define $\underline{s}_{i+1}^x$ $(\overline{s}_{i+1}^x) = \inf x^{-1}(a_{i+1})$ $(\sup x^{-1}(a_{i+1}))$. By definition $\underline{s}_{i+1}^y = \overline{t}_i^y$.

Suppose first that $\underline{t}_i^x \leq \underline{t}_i^y$ and $\overline{t}_i^x > \overline{t}_i^y$ by the inductive hypothesis. Note that by monotonicity, $\underline{s}_{i+1}^x \geq \overline{t}_i^x > \overline{t}_i^y = \underline{s}_{i+1}^y$. Now assume towards a contradiction that $\overline{s}_{i+1}^y \leq \overline{s}_{i+1}^x$. By incentive compatibility,

$$\begin{aligned}
& L^A(a_i, \overline{t}_i^x | x) \leq L^A(a_{i+1}, \overline{t}_i^x | x) \\
\iff & (a_i - \overline{t}_i^x)^2 - \rho R(\underline{t}_i^x, \overline{t}_i^x) \leq (\overline{t}_i^x - a_{i+1})^2 - \rho R(\underline{s}_{i+1}^x, \overline{s}_{i+1}^x) \\
\implies & (a_i - \overline{t}_i^x)^2 - \rho R(\underline{t}_i^y, \overline{t}_i^x) \leq (\overline{t}_i^x - a_{i+1})^2 - \rho R(\overline{t}_i^x, \overline{s}_{i+1}^y) \\
\implies & (a_i - \overline{t}_i^y)^2 - \rho R(\underline{t}_i^y, \overline{t}_i^y) < (\overline{t}_i^y - a_{i+1})^2 - \rho R(\overline{t}_i^y, \overline{s}_{i+1}^y) \\
\implies & L^A(a_i \overline{t}_i^y | y) < L^A(a_{i+1}, \overline{t}_i^y | y).
\end{aligned}$$

The first implication decreases the reputation on the RHS because by hypothesis $\overline{s}_{i+1}^x \geq \overline{s}_{i+1}^y$ and $\underline{s}_{i+1}^x \geq \overline{t}_i^x$ and increases the reputation on the LHS because $\underline{t}_i^x \leq \underline{t}_i^y$. The second implication invokes condition M^* moving the middle type from $\overline{t}_i^x$ down to $\overline{t}_i^y$. The last line represents a contradiction to incentive compatibility under y. This means that $\overline{s}_{i+1}^y > \overline{s}_{i+1}^x$. Since $y^{-1}(a_{i+1})$ is a nondegenerate interval, this means that by Claim 21, $a_{i+1} = \max A_i \cap y(T)$ and so $\tilde{A}_i$ must be a singleton and with $\overline{s}_{i+1}^x = \overline{t}_{i+1}^x$ and $\overline{s}_{i+1}^y = \overline{t}_{i+1}^y$ and $\underline{s}_{i+1}^x = \underline{t}_{i+1}^x$ and $\underline{s}_{i+1}^y = \underline{t}_{i+1}^y$. This completes the inductive step for this case.

Now suppose instead that $\underline{t}_i^x \geq \underline{t}_i^y$ and $\overline{t}_i^x < \overline{t}_i^y$. Note that by definition again, $\underline{s}_{i+1}^y = \overline{t}_i^y$. Consider first $\tilde{a} \equiv x(\overline{t}_i^x) < a_{i+1}$, i.e. y skips over action $\tilde{a}$. By Claim 20 and the fact that $\tilde{a}$ is not used under y we have

$$(a_i - \overline{t}_i^y)^2 + \rho(\overline{t}_i^y - R(\underline{t}_i^y, \overline{t}_i^y)) \leq (\tilde{a} - \overline{t}_i^y)^2, \text{ and} \tag{56}$$

$$(a_{i+1} - \overline{t}_i^y)^2 + \rho(\overline{t}_i^y - R(\underline{s}_{i+1}^y, \overline{s}_{i+1}^y)) \leq (\tilde{a} - \overline{t}_i^y)^2. \tag{57}$$

Define $\tilde{s} \equiv \sup x^{-1}(\tilde{a})$. I will first show that $\tilde{s} > \overline{t}_i^y$. Suppose not, i.e., $\tilde{s} \leq \overline{t}_i^y$. Then by incentive compatibility,

$$\begin{aligned} & L^A(a_i, \overline{t}_i^x | x) = L^A(\tilde{a}, \overline{t}_i^x | x) \\ \iff & (a_i - \overline{t}_i^x)^2 - \rho R(\underline{t}_i^x, \overline{t}_i^x) = (\tilde{a} - \overline{t}_i^x)^2 - \rho R(\overline{t}_i^x, \tilde{s}) \\ \implies & (a_i - \overline{t}_i^x)^2 - \rho R(\underline{t}_i^y, \overline{t}_i^x) \geq (\tilde{a} - \overline{t}_i^x)^2 - \rho R(\overline{t}_i^x, \overline{t}_i^y) \\ \implies & (a_i - \overline{t}_i^y)^2 + \rho(\overline{t}_i^y - R(\underline{t}_i^y, \overline{t}_i^y)) > (\tilde{a} - \overline{t}_i^y)^2 \end{aligned}$$

The first implication uses the hypothesis that $\tilde{s} \leq \overline{t}_i^y$, so that the reputation on the RHS increases, and that $\underline{t}_i^x \geq \underline{t}_i^y$ so that the reputation on the LHS is lower. The second implication invokes condition (M^*) moving the middle type from $\overline{t}_i^x$ up to $\overline{t}_i^y$. The last line contradicts (56) above.

Thus $\tilde{s} > \overline{t}_i^y$. Since by monotonicity $\tilde{s} \leq \underline{s}_{i+1}^x$ and $\overline{t}_i^y = \underline{s}_{i+1}^y$, we have $\underline{s}_{i+1}^y < \underline{s}_{i+1}^x$. Toward a contradiction, suppose that $\overline{s}_{i+1}^x \geq \overline{s}_{i+1}^y$. By incentive compatibility,

$$\begin{aligned} & L^A(\tilde{a}, \tilde{s} | x) \leq L^A(a_{i+1}, \tilde{s} | x) \\ \iff & (\tilde{a} - \tilde{s})^2 - \rho R(\overline{t}_i^x, \tilde{s}) \leq (a_{i+1} - \tilde{s})^2 - \rho R(\underline{s}_{i+1}^x, \overline{s}_{i+1}^x) \\ \implies & (\tilde{a} - \tilde{s})^2 - \rho R(\overline{t}_i^y, \tilde{s}) < (a_{i+1} - \tilde{s})^2 - \rho R(\tilde{s}, \overline{s}_{i+1}^y) \\ \implies & (\tilde{a} - \overline{t}_i^y)^2 < (a_{i+1} - \overline{t}_i^y)^2 - \rho R(\overline{t}_i^y, \overline{s}_{i+1}^y). \end{aligned}$$

The first implication uses the fact that $\overline{t}_i^y > \overline{t}_i^x$ to increase the reputation on the LHS, and that $\tilde{s} < \underline{s}_{i+1}^x$ and the hypothesis that $\overline{s}_{i+1}^x \geq \overline{s}_{i+1}^y$ to decrease the reputation on the RHS. The second implication invokes condition (M^*) moving the middle type from $\tilde{s}$ down to $\overline{t}_i^y$. The last line contradicts (57). This means that $\overline{s}_{i+1}^y > \overline{s}_{i+1}^x$. Since $y^{-1}(a_{i+1})$ is a nondegenerate interval, this means that by Claim 21, $a_{i+1} = \max A_i \cap y(T)$ and so $\tilde{A}_i$ must be a singleton and with $\overline{s}_{i+1}^x = \overline{t}_{i+1}^x$ and $\overline{s}_{i+1}^y = \overline{t}_{i+1}^y$ and $\underline{s}_{i+1}^x = \underline{t}_{i+1}^x$ and $\underline{s}_{i+1}^y = \underline{t}_{i+1}^y$. This completes the inductive step for this case.

Lastly suppose that $x(\overline{t}_i^x) = a_{i+1}$, i.e. $\underline{s}_{i+1}^x = \overline{t}_i^x$. Towards a contradiction suppose that $\overline{s}_{i+1}^x \leq \overline{s}_{i+1}^y$. Then by incentive compatibility

$$\begin{aligned} & L^A(a_i, \overline{t}_i^y | y) = L^A(a_{i+1}, \overline{t}_i^y | y) \\ \iff & (a_i - \overline{t}_i^y)^2 - \rho R(\underline{t}_i^y, \overline{t}_i^y) = (a_{i+1} - \overline{t}_i^y)^2 - \rho R(\overline{t}_i^y, \overline{s}_{i+1}^y) \\ \implies & (a_i - \overline{t}_i^y)^2 - \rho R(\underline{t}_i^x, \overline{t}_i^y) \leq (a_{i+1} - \overline{t}_i^y)^2 - \rho R(\overline{t}_i^y, \overline{s}_{i+1}^x) \\ \implies & (a_i - \overline{t}_i^x)^2 - \rho R(\underline{t}_i^x, \overline{t}_i^x) < (a_{i+1} - \overline{t}_i^x)^2 - \rho R(\overline{t}_i^x, \overline{s}_{i+1}^x) \\ \iff & L^A(a_i, \overline{t}_i^x | x) < L^A(a_{i+1}, \overline{t}_i^x | x). \end{aligned}$$

The first implication uses that $\underline{t}_i^x \geq \underline{t}_i^y$ which increases the reputation on the LHS and the hypothesis that $\overline{s}_{i+1}^x \leq \overline{s}_{i+1}^y$ which decreases the reputation on the RHS. The second implication invokes condition (M^*) moving the middle type down from $\overline{t}_i^y$ to $\overline{t}_i^x$. The last line contradicts incentive compatibility of x. This means that $\overline{s}_{i+1}^x > \overline{s}_{i+1}^y$. Since $\overline{s}_{i+1}^y \geq \overline{t}_i^y > \overline{t}_i^x$, $x^{-1}(a_{i+1})$ is a nondegenerate interval which means that $\{a_{i+1}\}$ is an isolated action in $x(T) = \tilde{A}$, and so by the form of simple delegation sets, for j such that $\tilde{A}_i = A_j \cap y(T)$, A_j is a singleton, which then implies that $\tilde{A}_i$ is a singleton. This means that $\overline{t}_{i+1}^x = \overline{s}_{i+1}^x$, $\overline{t}_{i+1}^y = \overline{s}_{i+1}^y$, $\underline{t}_{i+1}^x = \underline{s}_{i+1}^x$, and $\underline{t}_{i+1}^y = \underline{s}_{i+1}^y$ completing the inductive step in this case and proving the claim.

Now take the highest action $y(M) \in \tilde{A}_n$. By Claim 22, each $\tilde{A}_i$ is a singleton so this is well defined. Note that if $\underline{t}_n^x < \underline{t}_n^y$, then by Claim 22 $\overline{t}_n^x > \overline{t}_n^y$ which contradicts that $\overline{t}_n^y = M$. This means that $\underline{t}_n^x \geq \underline{t}_n^y$ and by Claim 22 $\overline{t}_n^x < \overline{t}_n^y$. Let $y(M) \equiv a_n$ and $x(\overline{t}_n^x) \equiv a_{n+1}$, i.e. a_{n+1} is off-path under y. By incentive compatibility,

$$\begin{aligned}
& L^A(a_n, \overline{t}_n^x | x) = L^A(a_{n+1}, \overline{t}_n^x | x) \\
\iff & (a_n - \overline{t}_n^x)^2 - \rho R(\underline{t}_n^x, \overline{t}_n^x) = (a_{n+1} - \overline{t}_n^x)^2 - \rho r_x(a_{n+1}) \\
\implies & (a_n - \overline{t}_n^x)^2 - \rho R(\underline{t}_n^y, \overline{t}_n^x) \geq (a_{n+1} - \overline{t}_n^x)^2 - \rho R(\overline{t}_n^x, M) \\
\implies & (a_n - M)^2 + \rho(M - R(\underline{t}_n^y, M)) > (a_{n+1} - M)^2 \\
\iff & L^A(a_{n+1}, M | y) > (a_{n+1} - M)^2.
\end{aligned}$$

The first implication uses that $\underline{t}_n^x \geq \underline{t}_n^y$ which decreases the reputation on the LHS and that $M \geq \sup x^{-1}(a_{n+1})$ which increases the reputation on the RHS. The second implication invokes condition (M^*) moving the middle type $\overline{t}_n^x$ up to M.[41] Notice also that $L^A(t'|y) - (a_{n+1} - t')^2 + \rho(t' - \tilde{R})$ is strictly decreasing in t' by Lemma 8 and the fact that $a_{n+1} > y(t)\ \forall t$. Thus type M prefers a_{n+1} to their allocation under y for some $\tilde{R}$ reputation and has this preference for a strictly larger set of $\tilde{R}$ reputations than any other type. Thus the D1 refinement says that the off path belief puts 0 probability on any type $t \neq M$ following action a_{n+1}. But then the last line in the display is a contradiction to D1 incentive compatibility, completing the argument.

Q.E.D.

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[41] When M is infinity the associated expressions should be evaluated as the limit as $M \to \infty$.