Reputational Delegation*

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Abstract

I study how a principal should delegate to an agent with career concerns. The agent and principal are assumed to have aligned intrinsic incentives. However, the market observes the chosen action, and rewards the agent more, the higher it perceives his (private) type to be. This "reputational bias" has many similarities to the classic "material bias" studied in the communication literature, for instance, both can induce the same cheap talk equilibrium sets. However, I show that it is *always* optimal to impose a floor on the set of available actions in reputational delegation. This is in stark contrast to delegation to an agent with a material bias, where it is *never* optimal to restrict the agent's flexibility to take low actions. I specialize to the exponential family of distributions to show that offering flexibility to high types (i.e. full separation) is optimal. This result uses a recursive approach novel to communication problems.

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1. Introduction

Delegation is useful when the individual with decision rights is not the individual with the most relevant information. For example, school administrators delegate grading to teachers who better know the true ability of each student, private equity investors cede investment decisions to the fund's general partners, and law firm partners allow associates to decide how many billable hours to put into each case. The key tension is that the agent – the individual with more information – may be biased relative to the principal – the individual with decision rights.

A focal case is when the agent is biased toward higher decisions relative to the principal, e.g. a teacher wants to give students higher grades than are warranted by their performance, a venture capitalist wants to invest more than necessary in each entrepreneur, and an associate wants to put excess hours into each case. There are many reasons that such biases may arise. For example, the teacher may like their students and want them to do well, the venture capitalist may face limited downside risk relative to the end investors, and the associate's bonus may be positively tied to their hours worked. These agents have a *material bias*: an exogenous preference for higher actions than the principal would take with the same information. The large literature on delegation initiated by Holmstrom (1984) studies how a principal should delegate to an agent with a material bias.

Alternatively, the agents could have a *reputational bias* towards choosing higher actions. This arises when situations that warrant a higher action are associated with those in which the agent has higher ability. For example, better teachers educate their students to perform better and therefore tend to give higher grades, better venture capitalists find more promising entrepreneurs who warrant larger investments, and more dedicated associates put more work into each case. In each of these examples the agent has an incentive to choose an even higher action in order to signal higher ability. In contrast to a material bias, a reputational bias is endogenous.

There is often a subtle distinction between when an agent's incentives induce a reputational bias and when they induce a material bias. As a result, many economic interactions that motivate classic delegation studies could alternatively involve a reputationally biased agent. For example, workers may be biased towards working excessive hours in order to earn higher bonuses. Law firm associates receive a bonus if they work more than some predetermined amount of hours within a year, while investment banking analysts receive a bonus that depends on their manager's subjective view of their performance. In the law firm, bonuses induce a material bias towards working more hours: there is an exogenous relationship between hours worked and bonuses. Conversely, in the investment bank, bonuses induce a reputational bias towards more hours: an analyst's incentives depend endogenously on how many hours less-dedicated analysts work, and his desire to separate from them in the eyes of the manager. This motivates the main questions of this paper: how should a principal delegate to an agent with a reputational bias, and how does this differ from that for an agent with a material bias?

My model is the same as the standard delegation model, except for the agent's preferences. It features a single principal who wants to match his chosen action to an agent's private type, both of which are positive real numbers. The agent wants to match the action to their type, that is, their material component is aligned with the principal. However, the agent also benefits from giving the market – which only observes the action he chooses – the impression that he is a higher type. The principal delegates a subset of actions to the agent. I focus on the direct mechanism approach, in which the principal optimizes over incentive compatible *allocations*, i.e. mappings from agent types to actions.[1]

The main takeaway from the material delegation literature is that the principal should "cap against the agent's bias" (see Holmstrom (1984), Alonso and Matouschek (2008), Amador and Bagwell (2013)). This has two dimensions: (i) the agent is restricted from taking high actions, and (ii) the principal should never restrict the agent's flexibility to take low actions.[2] In contrast, Theorem 1 shows that in reputational delegation the principal always wants to impose a floor. That is, the principal pools some lower set of agent types by restricting them to take an inefficiently high action.

Theorem 1 leaves open whether a floor, i.e. pooling low types, is beneficial, or if it is simply pooling that is beneficial or necessary, as in cheap talk (Crawford and Sobel (1982)). Answering this question requires more detail about the optimal delegation set, for which I specialize to the exponential family of type distributions. Theorem 2 shows that the optimal allocation is eventually separating, i.e. above some type, the agent perfectly reveals his type with his chosen action. I also show that for exponential distributions close to the uniform limit, the optimal delegation set involves a floor and then full separation (flexibility) above some action. Methods for solving the material delegation problem, e.g. those in Kleiner et al. (2021) and Amador and Bagwell (2013), do not work in the reputational delegation model. Instead, I solve the principal's problem using recursive methods that

[1] One difference from the material delegation framework is that following delegation, the agent and the market play a Bayesian game with potentially multiple equilibria. In Section 5 I show that under a regularity assumption on the distribution of types, there is a unique equilibrium that survives the D1 refinement, and that this equilibrium uses all delegated actions. In this sense, the principal can uniquely implement any incentive compatible allocation by delegating the set of on-path actions.

[2] Even when interval delegation is not optimal in the material delegation model, it is always optimal to give the lowest type his ideal action and also make all lower actions available.

are (to my knowledge) novel in communication models.

To interpret the main results, note that in the exponential model, the optimal delegation set is of the form $\{a_0\} \cup [\underline{a}, \infty)$, where $0 < a_0 < \underline{a}$. In general, setting a floor means the principal restricts flexibility at the bottom to some isolated action a_0. In contrast, under material delegation with some basic regularity assumptions, it is optimal to fully delegate in the exponential model.[3] In the three economic examples above, setting a floor can be interpreted as: pooling all low performing students on an F grade, setting some minimum investment for contracted entrepreneurs, or forcing associates to put at least a fixed amount of hours into each case.

While there are many alternative explanations for these delegation policies, the intuition in the current model comes from the optimal floor reducing distortionary signaling incentives for high ability agents. That is, setting an optimal floor reduces incentives to teach to the test in order to signal a quality education, to over-invest in order to signal prescience, or to put in inefficient hours in order to signal diligence. The mechanism behind this derives from the endogeneity of the reputational bias, i.e. the agent's incentive to take a higher action depends on the allocation. This opens up a channel for the principal to sacrifice efficiency for a set of agent types in order to decrease the effective bias on some other group of types. Indeed, this is the intuition for Theorem 1: the principal sacrifices loss on low types in order to improve the allocation for higher types. Specifically in the exponential model, the types that take actions $[\underline{a}, \infty)$ under the optimal delegation set are better matched than would take these actions under full delegation.

The methodology behind both main results exploits the following recursive structure. Consider an arbitrary incentive compatible allocation x with a threshold at some agent type t, i.e. all types below t get different actions than all types above t. I refer to the set of *continuing allocations* above t as those that would maintain incentive compatibility when juxtaposed with x below t. This set depends on x only through the loss that type t experiences. Correspondingly, I refer to $V(t, u)$ as the minimized continuation loss for the principal among the set of continuing allocations given that type t experiences loss u from the allocation below. If x is optimal, naturally, it must minimize loss for the principal among all other IC allocations with a threshold t. With the definition of V, this means that x must minimize the loss below t plus the minimized continuation loss after t. For a low enough threshold type t, the difference in loss below type t across allocations is relatively small. Thus, the optimal choice for the allocation below type t comes down to which one

[3] There is no optimal cap in the standard quadratic loss material delegation model with an exponential type distribution. Indeed, this is generically true for many unbounded type distributions. In general an interval of actions near 0 is in every optimal delegation set.

induces the lowest continuation loss above t.

The key to answering this question is Proposition 1 – termed *the alignment principle* – which says that $V(t,u)$ is strictly increasing in u. That is, increasing the agent's loss at type t increases the principal's minimized continuation loss above type t. I show that an allocation that pools some set of types $[0,t]$, i.e. implements a floor, decreases the loss for type t relative to allocations which distinguish types in this interval. The reasons are twofold. First, separating agent types by giving them different actions causes distortion (the distance between the chosen action and type) to accumulate very quickly for low types where distortion is small. To see this, note that the principal must excessively increase the action between separating types in order to deter the agent from seeking a higher reputation. When distortion is small, the action needs to increase by a lot in order to provide this deterrence, hence distortion accumulates quickly. Second, the lowest action, i.e. floor, is special in that it does not need to respect any lower types' incentive constraints. This means that setting a non-trivial floor can bypass the first issue.

Specializing to the exponential model in Section 4 makes the set of continuing allocations and the minimized continuation loss independent of the initial type t. In addition, the continuation loss admits a recursive structure: the first pooling interval pins down both the loss for the principal on this first interval and the initial loss for the first threshold. Proposition 2 shows that a set of continuation losses that solve the associated Bellman equation must be optimal. I use this result to establish Theorem 2, i.e. that separating is eventually optimal. The intuition reverses that for the optimal floor: (i) for high types where distortion is already large, separating types increases loss very gradually relative to pooling, and (ii) unlike with the floor, the action for any other pooled set has to respect the incentive constraint for some lower type.

1.1. Related Literature

The delegation literature was initiated by Holmstrom (1984) which, assuming interval delegation, finds that it is optimal to cap against the agent's bias. Alonso and Matouschek (2008) and Amador and Bagwell (2013) study generalizations of this standard model and reach interval delegation as a conclusion under palatable assumptions on the preferences and distribution of types. Under some assumptions, Kovac and Mylovanov (2009) shows that these features are robust to allowing for stochastic mechanisms. Dessein (2002) and Melumad and Shibano (1991) also study the original model and compare delegation to alternative communication protocols.

A number of papers study different delegation technologies without changing the material bias of the agent. Krishna and Morgan (2008) and Ambrus and Egorov (2017) study the delegation model with transfers or money burning. Armstrong and Vickers (2010) study an agent who has private information over which choices are available. Frankel (2014) studies a delegation model with many decisions and takes a worst case approach. Halac and Yared (2020) adds the possibility for the principal to verify the agent's type at a cost.

My paper is also related to the large literature dealing with career concerned agents initiated by Holmström (1999), which studies a model where the outcome can be observed. My paper fits better into an alternative strand which assumes that only the agent's choice can be observed. Examples like Morris (2001), Prendergast and Stole (1996), and Scharfstein and Stein (1990). Visser and Swank (2007), Moscarini (2007), Kartik and Van Weelden (2018) all integrate career concerns into cheap talk models. A closely motivated set of papers is Ottaviani and Sorensen (2006a) and Ottaviani and Sorensen (2006b) which attempt to compare predictions in cheap talk when the agent is motivated by career concerns vs. material concerns.

Similarly, the incentives in the current reputational communication model are related to those in classic costly signaling models, e.g. in Spence (1973). Frankel and Kartik (2019) study a signaling model whose preferences map closer to those in the current model. Many studies ask various design questions in this framework. Zubrickas (2015) and Dubey and Geanakoplos (2010) ask how to optimally pool test scores into grades (the latter examines when test scores are random from the perspective of the student). Relatedly, Saeedi and Shourideh (2020) Hopenhayn and Saeedi (2019), Hörner and Lambert (2020) study how an intermediary can commit to information disclosure policies to increase the quality of goods produced or effort taken.[4] One ostensibly close exercise is that in Onuchic and Ray (2021), which looks at restricting the choice set of education levels avalable in a costly signaling problem. However, their focus is on issues of a non-common prior between the student and university. They also impose a linear utility for the student over education choices which is downward biased relative to the university, and that the lowest education level must be in the choice set, so the tradeoffs highlighted in this paper are not present.[5]

Kartik (2009) and the costly lying literature also studies signaling tradeoffs similar to that in the current paper. Roughly, the costly message is mapped to the action and the receiver's response to the message is mapped to the reputation. My main results speak to how a designer should limit the set of messages available to the sender in these models.

In broad strokes, the main intuition behind the optimal floor has echoes in the eco-

[4] See Lizzeri (1999) and Ostrovsky and Schwarz (2010) which study related questions without the moral hazard component.

[5] For example, the separating allocation in their model is independent of the current distortion, removing the key point that distortion accumulates especially quickly for low types.

nomic theory literature going back to Maskin and Riley (1984) and Rothschild and Stiglitz (1976). The reason why a low value or low risk buyer is sometimes excluded in contracting problems is because respecting these buyer's incentive constraints negatively impacts profits on high value buyers. More recent examples include Fuchs and Skrzypacz (2015), which shows that an initial trade subsidy increases welfare over laissez-faire policies, and Krishna and Morgan (2008) and Karamychev and Visser (2017) who show that using transfers/money burning to incentivize the correct action for high types is sub-optimal.

The use of recursive optimization methods in communication problems is apparently novel. However, the recursive structure of communication problems with monotonic allocations is used to construct cheap talk equilibria in Crawford and Sobel (1982). Deimen and Szalay (2019) specializes to a family of Laplace distributions to simplify this procedure. This is analogous to how specializing to the exponential distribution over the types reduces the dimensionality of finding the optimal delegation set.

2. Model and Preliminary Results

2.1. Setup

Overview There is a principal, an agent, and an outside observer or market. The agent has private information about his type $t \in T \equiv [0, M]$ where $M \in [0, \infty)$. There is a common prior type distribution with density $f : T \to \mathbb{R}^+$ and associated measure F. I assume the density is bounded above and below, i.e. $\exists \overline{k} \geq \underline{k} > 0$ with $f(t) \in [\underline{k}, \overline{k}]$ $\forall t \in T$. The principal has control over an action $a \in A \equiv \mathbb{R}^+$ and delegates a subset of available choices to the agent. The agent chooses an action from this delegation set. The market then observes the chosen action and updates their belief about the agent's type.

Preferences I represent preferences in terms of losses instead of utilities as it proves more convenient throughout. The principal's loss given action choice a and type t is given by $(a - t)^2$. The agent has two components to his preferences, a material component and a reputational component. The material component is the same as the principal's, i.e. there is no material misalignment. Given a belief $\mu \in \Delta T$ held by the market, an agent of type t has reputational loss given by $\rho(t - \mathbb{E}[t'|t' \sim \mu])$, where $\rho > 0$ is a positive constant weight. The reputational loss is normalized (i.e. it does not affect equilibrium behavior) to be 0 if μ is degenerate on t. Given market belief μ, action a, and type t, the total loss of the agent is given by

$$(a-t)^2 + \rho(t - \mathbb{E}[t'|t' \sim \mu]).$$[6]

Delegation and Equilibrium Focus The principal delegates a set $\tilde{A} \subset A$. The agent's strategy is a mapping from types to distributions over actions in the delegation set: $\sigma : T \to \tilde{A}$.[7] The market forms beliefs over the agent's type given the chosen action, $\mu : \tilde{A} \to \Delta T$. An equilibrium given $\tilde{A}$ is a pair, σ, μ such that $\mu(a)$ is consistent with Bayes rule $\forall a \in \mathrm{Supp}(\sigma)$, and σ minimizes loss for the agent given reputations consistent with μ.

In general, there can be multiple equilibria for the same delegation set $\tilde{A}$.[8] However, many of these equilibria are sustained by unreasonably punitive off-path beliefs, e.g. beliefs that assume that all off-path actions are taken with certainty by $t = 0$. I instead focus on the equilibria satisfying the classic D1 refinement introduced by Cho and Kreps (1987).[9] In Section 5, I show that under a regularity condition on the type distribution, there is a unique equilibrium satisfying the D1 refinement for any delegation set $\tilde{A}$. Moreover, this unique D1 equilibrium uses all delegated actions, i.e. $\tilde{A} = \sigma(T)$.

This result justifies focus on the direct mechanism approach. The principal specifies an incentive compatible **allocation** $x : T \to A$, and uniquely implements it by delegating all used actions, i.e. $\tilde{A} = x(T)$. This implementation also allows one to ignore considerations of off-path beliefs. Outside of the D1 refinement, the proceeding analysis is still relevant if one focuses on the principal's preferred equilibrium given any delegation set. Until Section 5, I omit further mention of the delegation set and focus on the allocation $x : T \to A$.

Allocations Given an allocation $x : T \to A$, the market believes the action is chosen via the allocation. Thus, the market's belief over the type after observing action $a \in x(T)$ is simply the prior conditioned on $x^{-1}(a)$. This means that each allocation induces reputations given by $r_x(a) \equiv \mathbb{E}[t'|t' \in x^{-1}(a)]\ \forall a \in x(T)$. I refer to the realized reputation as $r^*_x(t) \equiv r_x(x(t))$. Given allocation x, the loss to the agent of type t from choosing action $a \in x(T)$, is denoted $L^A(a, t|x) \equiv (a-t)^2 + \rho(t - r_x(a))$. I refer to the realized loss for

[6] In Section 6, I discuss how the main results extend to more general material and reputational preferences. I also discuss how the results would change if the principal and agent were misaligned on their material preferences.

[7] Because of the quadratic loss material preferences, it is never optimal for the agent to mix between actions for a positive measure of types. Thus, the restriction to pure strategies is without loss.

[8] This equilibrium multiplicity is only a potential issue for the principal in "one direction": if there is an alternative equilibrium which uses more actions than the intended equilibrium σ, the principal can just shrink the delegation set to $\tilde{A} = \sigma(T)$. The real issue is alternative equilibria that use fewer actions than intended.

[9] See Section 5 for details.

type t as $L^A(t|x) \equiv L^A(x(t), t|x)$. The expected loss for the principal is denoted $L^P(x) \equiv \int_0^M (x(t') - t')^2 f(t')dt'$. Note that because the agent's preferences over market beliefs are linear, the expected reputational loss is 0 for any allocation x and $L^P(x) = \mathbb{E}[L^A(t|x)]$.

Incentive Compatibility The principal is restricted to choose incentive compatible (IC) allocations. Given the reputations r_x, the agent of each type t must be incentivized to choose their allocated action $x(t)$. That is, $x(t)$ is an equilibrium of the Bayesian game between the agent and the market given choice set $x(T)$. An allocation $x : T \to A$ is incentive compatible on T if

$$L^A(t|x) = \min_{a \in x(T)} L^A(a, t|x) \quad \forall t \in T. \tag{1}$$

Let the set of allocations satisfying (1) be $IC(T)$.[10] The principal seeks to minimize their loss over all incentive compatible allocations. That is, the principal solves

$$\inf_{x \in IC(T)} L^P(x). \tag{2}$$

Discussion One can interpret the reputational concerns of the agent in a couple different ways. The first interpretation is implied by the setup above: the reptutational concern is a reduced form for the agent being compensated in the future based on the market's belief about his type. For example, as stated in the introduction, the type could be correlated with ability and so the market's belief could capture future hiring opportunities.

The reputational concern could also capture the expected gains in future repeated identical interactions with the same principal who lacks dynamic commitment power and can choose to replace the agent. To be succinct, I have not modeled the principal as valuing higher type agents. However, if the principal's loss were instead given by $(a - t)^2 - t$, this would induce an incentive to replace low type agents, and clearly not change the optimal allocation. Through all possible interpretations, it is very important that only the action and not the loss, nor the type of the agent is observed by the party whose beliefs the agent values.

The model differs from the material delegation framework only through the agent's preferences. The exogenous material bias in the standard model has been replaced by an endogenous reputational bias. The reputational bias still leads the agent to prefer higher

[10] It will also prove useful to correspondingly refer to the set of allocations that are incentive compatible on some subset of types $S \subset T$ as $IC(S)$.

actions than the principal. However, the extent to which the agent and principal are misaligned depends on the allocation. To see why the reputational term leads the agent to be upward biased relative to the principal, consider two actions $a_1 < a_2$ in the range of some incentive compatible allocation x. As will be made precise in Lemma 1 below, higher actions must be allocated to higher types, meaning that $r_x(a_2) > r_x(a_1)$. This implies that if the agent of type t is indifferent between a_1 and a_2, the material loss must actually be lower for a_1, i.e. $(a_1 - t)^2 < (a_2 - t)^2$. This is equivalent to saying that the principal strictly prefers to allocate a_1 rather than a_2 to type t. As is discussed in Section 6, preserving this upward biased condition is essential to considering extending the results to more general preferences.

It may seem strange to set up of a comparison of these two models that have fundamentally different agent preferences. However, material bias communication models have a lot in common with the reputational bias model in this paper. Section 6 discusses how under cheap talk, the canonical material bias model has the same set of equilibria as that for the current reputational bias model. In addition any incentive compatible allocation in the current reputational model is an IC allocation in *some* material bias delegation model, and vice versa. The key point, as will be seen in the proceeding analysis, is that the *set* of IC allocations is always different between the current reputational delegation model and *any* material delegation model.

2.2. Preliminaries

I first characterize incentive compatible allocations. For any allocation x let $J_x \subset T$ be its set of discontinuities.

Lemma 1. *An allocation $x \in IC(T)$ if and only if*

1. *x is increasing.*
2. *J_x is a countable nowhere dense set. Let $\underline{t}, \bar{t} \in J_x$ such that $(\underline{t}, \bar{t}) \cap J_x = \emptyset$. $\forall t \in (\underline{t}, \bar{t})$, either*
   (a) $x'(t) = 0$ *or* *(pooling)*
   (b) $x'(t) = \frac{\rho}{2(x(t)-t)}$ *(separating)*
3. *$\forall t \in J_x$,*
$$\lim_{t' \to t^+} L^A(x(t'), t|x) = \lim_{t' \to t^-} L^A(x(t'), t|x).$$

Given the characterization, I equate each $x \in IC(T)$ with its right continuous counterpart.[11] Lemma 1 is very similar to the characterization of incentive compatibility in material

[11] This changes the allocation on a measure zero set of types and thereby does not affect the principal's loss.

delegation: it is identical to lemma 1 in Alonso and Matouschek (2008) aside from point (2b).

First, all IC allocations are monotonic. The reason is that for any allocation x, $L^A(a,t|x)$ is strictly submodular in (a,t). This monotonicity means that in each IC allocation, each action is associated with an interval of types, and so the reputations are the associated expectations on these intervals. That is $r_x(a) = \mathbb{E}[t'|\underline{t} \leq t' \leq \bar{t}]$ for some $\underline{t} \leq \bar{t}$. Because of this feature, it will be useful to notate $R(\underline{t},\bar{t})\mathbb{E}[t'|\underline{t} \leq t' \leq \bar{t}]\ \forall \underline{t} \leq \bar{t}$. The second part of the result says that any $x \in IC(T)$ segments the type space into countably many intervals on which the allocation is either constant, or a solution to the differential equation in (2b). I refer to these two types of allocations as pooling and separating respectively. While this guarantees local incentive compatibility on each interval, the last part of the lemma guarantees local incentive compatibility at the endpoints.

As mentioned above, the main difference between Lemma 1 and the analogous characterization in material delegation is the behavior of the separating allocation. A separating allocation on a given interval $(\underline{t},\bar{t})$ is defined by the property that the action reveals the type, i.e. $x^{-1}(x(t)) = t$ or $r_x^*(t) = t\ \forall t \in (\underline{t},\bar{t})$. In this case incentive compatibility means that,

$$t \in \text{argmin}_{t' \in T}(x(t') - t)^2 + \rho(t - t').$$

Taking first order conditions gives the differential equation in (2b).

Figure 1 illustrates an example of an IC allocation with $|J_x| = 3$. The left panel displays the action as a function of the type while the right panel displays the corresponding loss for the agent. A few observations are worth noting now. First, the action jumps whenever the allocation switches from pooling to pooling, separating to pooling, or pooling to separating, because in each of these cases there is also a jump in the corresponding reputations. This is in contrast to material delegation wherein the allocation only jumps between two pooling intervals. Second, the loss of the agent is continuous in the type and almost everywhere differentiable. Third, and most importantly, the separating allocation is different than that for material delegation; in particular, the separating loss to the agent is increasing in the type, while in material delegation the agent gets his ideal action on separating intervals. The separating allocation will be important for the analysis and so it will prove useful to further explore (2b).

There are a continuum of increasing solutions to (2b) pinned down by the initial condition. Let $d_u(t)$ solve (2b) with initial condition $d_u(0)^2 = u$. Let $D_u(t) \equiv (d_u(t) - t)^2$ be the separating loss given that type $t = 0$ experiences loss u. While $d_u(t)$ does not admit an explicit representation, key properties are derived below. The separating loss for various

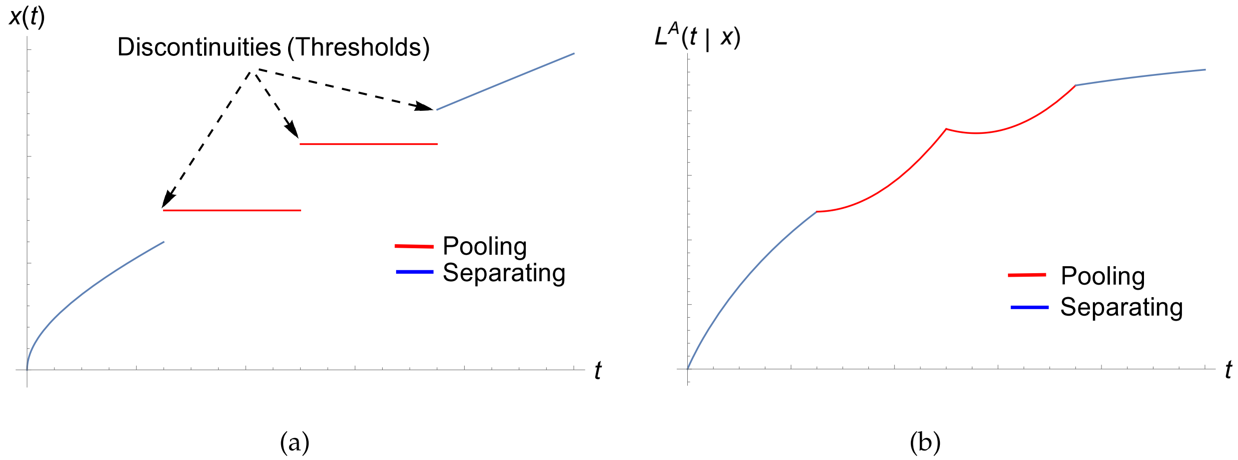


Figure 1: An Incentive Compatible Allocation

initial conditions is displayed in Figure 2.

Lemma 2. *Properties of the separating allocation:*

1. *If $u \leq (\geq)\rho^2/4$ then the loss of the agent $D_u(t)$, is increasing and concave (decreasing and convex) in t.*

2. $\forall u \geq 0$, $\lim_{t\to\infty} D_u(t) = \rho^2/4$.

3. *$\forall t$, $D_u(t)$ is strictly increasing in u.*

The first two points report that the separating loss monotonically asymptotes to $\rho^2/4$. At this loss, increasing the action one to one with the type exactly deters the agent from seeking a higher reputation, so loss remains constant. The third point says that the separating allocation is increasing in the initial loss u. This means that the principal's optimal separating allocation is given by $d_0(t)$ and is associated with the lowest curve in Figure 2.[12]

I conclude this section by asserting that a minimum to (2) exists.

Lemma 3. *There exists an allocation $x^* \in IC(T)$ that minimizes $L^P(x)$ across all $x \in IC(T)$.*

3. The Optimality of a Floor

Theorem 1 (Optimality of a Floor). *$\exists K > 0$ such that for every solution x^* to (2), $\exists F \geq K$ with $x^*(t) = F$ $\forall t \in [0, F]$.*

[12] The separating allocation is related to the separating equilibrium in lying cost models. Specifically (2b) is related to the differential equation in Lemma 1 in Kartik (2009).

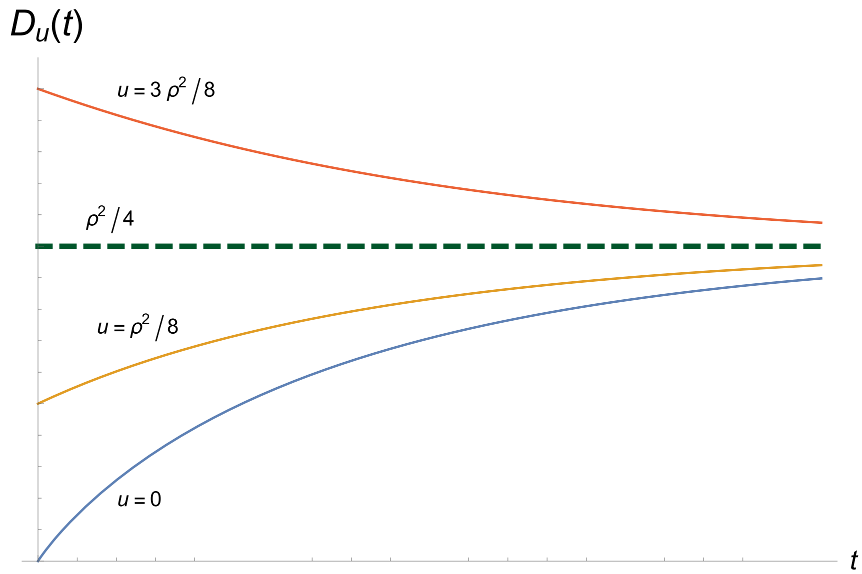


Figure 2: Separating Allocations

The main result says that it is optimal to restrict the agent's flexibility for low actions in any optimal allocation. Specifically, there is a uniform bound $0 < K \leq x^*(0)$ for any optimum x^*. An implication of this, is that the optimal allocation also pools a non-trivial first interval of types. This is because (as will be clarified below), it is always optimal to set the minimum action lower than the highest type to which it is allocated to, i.e. $x^*(0) < \sup(x^{*-1}(0))$. Another implication is that the floor action is isolated within the delegation set, i.e. $\exists\delta > 0$ such that for every optimal allocation $[0, x^*(0)+\delta] \cap x^*(T) = \{x^*(0)\}$. This is because the next highest action under x^* will have a discontinuously higher reputation than $x^*(0)$, so incentive compatibility necessitates a corresponding jump in the action as well. As made precise in Section 6, these features are unique to delegation with a reputational bias; in delegation to an agent with an upward material bias, it is never optimal to restrict flexibility at the bottom of the type distribution.

It is of course possible to construct IC allocations that violate the condition of Theorem 1. An example is the allocation in Figure 1. As the action for type $t = 0$ is $a = 0$, such alternative allocations perform better than any optimal allocation for low types. Indeed, any optimal allocation sacrifices some loss on low types by setting a floor in exchange for lowering the loss on higher types.

The argument behind this derives from the following thought experiment: given a threshold type $t > 0$ and that the principal is choosing the optimal allocation for types above t, how should the principal select the allocation below t in order to minimize total loss? In general, this consideration involves two terms: the loss below t, and the loss above t. In the absence of a floor though, the threshold t can be taken arbitrarily small, and so the loss below t is unimportant. Thus, in designing the allocation below this threshold t, the important consideration is how this affects the loss above t.

While the proof sketch is in Subsection 3.2, the basic intuition follows two steps: (i) pooling types below t reduces type t's loss relative to any other allocation below t, and (ii) the loss of type t is a proxy for the principal's loss above type t. Point (ii) holds generally and is termed the alignment principle in the next subsection. Some intuition for point (i) can be gleaned by inspecting the separating allocation. The idea is that distinguishing between agent types that experience small losses requires dramatic increases in the action. Consider (2b) which characterizes the separating allocation. In this extreme case in which the principal distinguishes all types, the action increases infinitely fast as distortion – $(x(t)-t)$ – becomes small. Such dramatic increases are necessary because the agent's material loss near his efficient outcome is arbitrarily small relative to the reputational benefit of claiming to be a higher type. This makes "ignoring" the IC constraints of these low types by imposing a floor especially attractive.

3.1. Continuing Allocations and the Alignment Principle

Consider some type $t \in T$, an allocation $x \in IC([0,t])$ below t, and another allocation $y \in IC([t,M])$ above t. Under what conditions can one join these allocations together to form an incentive compatible allocation? That is, when is the allocation defined by

$$z(t') \equiv \begin{cases} x(t') & t < t \\ y(t') & t \geq t \end{cases}$$

an element of $IC(T)$. The answer is given by point 3 of Lemma 1: the agent of type t must be indifferent between x and y. That is, the only information about x needed to specify y is the loss that type t experiences under x. Motivated by this, Let $y : [t,M] \to A$ be a **continuing allocation** at (t,u) if $y \in IC([t,M])$ and $L^A(t|y) = u \geq 0$. Denote the set of these continuing allocations as $C(t,u)$.

Define the **minimized continuation loss** as

$$V(t,u) \equiv \frac{1}{F([t,M])} \inf_{y \in C(t,u)} \int_t^M (y(t') - t')^2 f(t') dt'. \tag{3}$$

Note that if one solves (3) for every (t,u) then this greatly simplifies finding the optimal allocation. To see this, note that the solution to (2) is the solution to

$$\inf_{a_1,t_1} \int_0^{t_1} (a_1 - t')^2 f(t')dt' + F([t_1, M])V(t_1, (a_1 - t_1)^2 + \rho(t - R(0, t_1))). \tag{4}$$

That is, the principal only needs to optimize over the first action and first threshold. These choices pin down the agent's loss at the first threshold and thereby associated minimized continuation loss. This is the method used in Section 4. For now I focus on how $V(t, u)$ changes with the initial loss u.

An illustrative example of a continuing allocation is the separating continuing allocation at (t, u) given by $d_u(t' - t) + t\ \forall t' > t$. As derived in Lemma 2 point 3, the principal does better with the separating continuing allocation when the initial loss u is lower. The next proposition shows that this property extends to the minimized continuation loss.

Lemma 4. *$\forall u \geq 0$ and $t \in T$, there exists a solution to* (3)*.*

Proposition 1 (The Alignment Principle)**.** *$\exists k > 0$ such that $\forall t \in T$, $\forall u > 0$, $\forall \varepsilon \leq u$,*

$$\frac{V(t, u) - V(t, u - \varepsilon)}{\varepsilon} > k.$$

The alignment principle says that the minimized continuation loss above type t is increasing in the initial loss of type t. Moreover, the magnitude of this change in continuation loss is uniformly bounded away from 0. The implication is that all else being equal, the principal should seek an allocation x below type t that minimizes type t's loss.

The intuition comes from the fact that the reputational concern induces a bias towards higher actions. Reconsider the piecewise allocation z above formed by joining x and y, and suppose that $L^A(t|x) < L^A(t|y)$, i.e. the principal has access to $C(t, L^A(t|x))$, but chooses an allocation in $C(t.L^A(t|y))$. Note that by monotonicity, $r_x^*(t) \leq t \leq r_y^*(t)$, i.e. x provides a lower reputation than y to type t. This means that in order to satisfy $L^A(t|x) < L^A(t|y)$, it must be that $(x(t) - t)^2 < (y(t) - t)^2$, i.e. material loss must be less from x than y for type t. This means that the principal would also prefer to assign $x(t)$ rather than $y(t)$ to type t; in other words their preferences are *aligned* at type t.

There are many ways the principal can take advantage of the slack introduced by this alignment. The method used in the proof is illustrated in Figure 3. Consider $u_1 > u_0 \geq 0$, and a continuing allocation x_1 at (t, u_1). The proof constructs a better continuing allocation at (t, u_0) by replacing x_1 on $[t, \tilde{t}]$ with the separating continuing allocation. Here, $\tilde{t}$ is the first type indifferent between this separating allocation and $x_1(\tilde{t})$ given the altered reputations. The left panel of Figure 3 illustrates these two continuing allocations, and the right panel shows the agent's loss for every type between t and the next threshold under x_1. there are two main conclusions from this figure: (i) because the principal's expected loss and the agent's expected loss are equal, this change decreases loss for the principal,[13] and (ii) the next threshold under x_1 has a lower initial loss under the new allocation than under x_1, which means that one can continue this construction across the entire type space. Doing so, yields an allocation with strictly lower loss.[14]

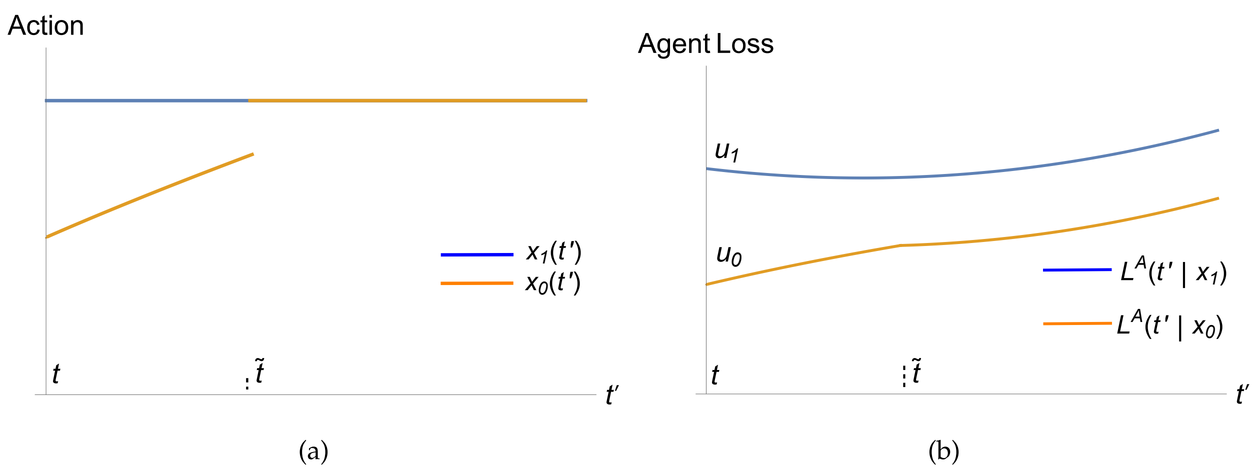


Figure 3: The alignment principle

3.2. Proof Sketch of Theorem 1

Consider that Theorem 1 does not hold. This means $\forall \varepsilon > 0$, there is an optimum x^* such that $x^*(0) < \varepsilon$. Let $\tilde{t}$ be the first threshold, i.e $r^*_x(0) = R(0, \tilde{t})$. Because of the alignment principle, it is dominant to choose $R(0, \tilde{t}) \leq x^*(0) \leq \tilde{t}$. Otherwise the principal could improve both the loss on $[0, \tilde{t}]$ and the loss of the first threshold $\tilde{t}$ by moving $x^*(0)$ into this interval. Thus $\tilde{t} < \varepsilon$ can be taken arbitrarily small as well.

It is useful to consider two cases based on whether the "next thresholds" under x^* can also be taken arbitrarily small or not.[15] Consider first that the next threshold $\tilde{s}$ is large. In this case, the action for $t \in [\tilde{t}, \tilde{s})$ is respecting the incentive constraint of $\tilde{t}$ at the benefit of decreasing the loss on $t \in [0, \tilde{t})$. However, because $\tilde{t}$ is small, the principal finds it beneficial to get rid of this first interval and freely optimize with respect to the new first action for $t \in [0, \tilde{s})$.

[13] The improvement is actually type by type.

[14] It is important that the rate of change is strictly positive rather than the weaker property that $V(u) > V(u - \varepsilon)$. Theorem 1 requires a first order change uniformly bounded away from 0.

[15] Formally, the division in the proof is as follows. Case 1: $\exists \tilde{s} \in J_{x^*}$ and $b > 0$ such that $(\tilde{t}, \tilde{s}) \cap J_{x^*} = \emptyset$ and $\tilde{s} > b$. Case 2: $\exists \tilde{s} \in J_{x^*}$ such that $\tilde{s} < \varepsilon$ and $\tilde{t}/\tilde{s} < \varepsilon$.

The alternative is that there exist some next threshold $\tilde{s}$ (not necessarily the adjacent one), that is small, i.e. $\tilde{s} < \varepsilon$, but large relative to $\tilde{t}$, i.e. $\tilde{t}/\tilde{s} < \varepsilon$. In this case, one can show that the loss for type $\tilde{s}$ is approximated by that of the minimized separating loss for $\tilde{s}$. Since the loss on $[0, \tilde{s})$ is small and second order for any (reasonable) allocation, this approximation and optimality of x^* implies that the separating allocation must minimize the loss of type $\tilde{s}$ among all allocations. Otherwise, the alignment principle would imply that the principal could improve her loss above $\tilde{s}$ by using some alternative continuing allocation. I use a floor to construct such an alternative allocation: let

$$z(t) \equiv \begin{cases} \tilde{s} & t < \tilde{s} \\ x^*_{\tilde{s},\tilde{u}}(t') & t \geq \tilde{s} \end{cases},$$

where $\tilde{u} \equiv \rho(\tilde{s} - R(0, \tilde{s}))$ and $x^*_{\tilde{s},\tilde{u}}$ is an optimal continuing allocation at $(\tilde{s}, \tilde{u})$. Figure 4 illustrates why z improves on x^*. The left panel illustrates how, under the uniform distribution and for small $\tilde{s}$, type $\tilde{s}$'s loss is lower under x than under the separating allocation. This comparison is general: for small $\tilde{s}$, the separating loss and pooling loss for small $\tilde{s}$ are approximated by $\rho\tilde{s}$ and $\rho\tilde{s}/2$ respectively. The right panel illustrates how z improves loss above $\tilde{s}$ relative to x^* when $x^*_{\tilde{s},\tilde{u}}$ is the separating continuing allocation. While the type by type comparison in loss is specific to the separating continuing allocation, the alignment principle shows that the comparison in expected loss holds generally.

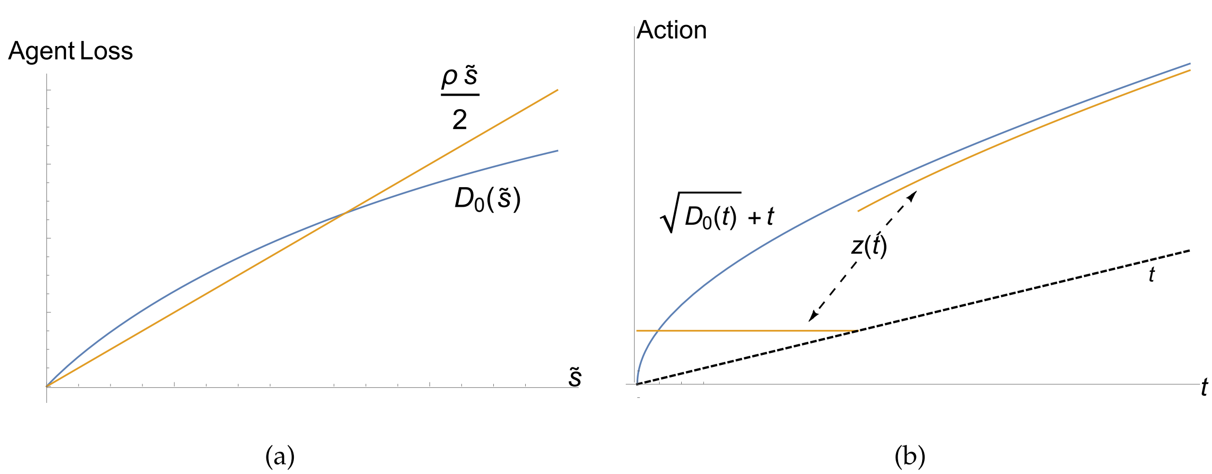


Figure 4: Improving on x^* Using a Floor.

4. The Exponential Model

The previous section showed that a floor is always optimal. The proof sketch makes use of the special properties of the first interval, i.e. that the corresponding action is unconstrained. However, another valid interpretation (at this point) is that pooling, i.e. discretely segmenting all types, is preferred in the reputational delegation framework.

Rebutting this alternative interpretation requires more details about the solution. However, standard methods used to solve the general material delegation model, e.g. those used in Amador and Bagwell (2013), and Kleiner et al. (2021), do not work in the reputational delegation framework.[16] Instead, I specialize to the exponential model in this section, and solve the problem using recursive methods. That is, for the remainder of this section, I assume that $T = [0, \infty)$ and $f(t) = \lambda e^{-\lambda t}$ $\forall t$ with $\lambda > 0$.[17] The main takeaway is that the separating allocation is "eventually" optimal, i.e. there exists some type above which the optimal allocation is separating. This means that pooling is not good per-se in reputational delegation, rather it is specifically optimal for low types.

The memorylessness property of the exponential distribution affords key simplifications. The set of continuing allocations and thereby the minimized continuation loss at (t, u) both do not depend on the initial type t. Given this fact, I normalize the initial type to 0, and denote a continuing allocation as $y : [0, \infty) \to A$, with $y \in IC(T)$, and $L^A(0|y) = u$. Furthermore, I abuse notation and write $C(t, u) \equiv C(u)$, $V(t, u) \equiv V(u)$, and $R(t_1, t_2) - t_1 \equiv R(t_2 - t_1)$. That is, the minimized continuation loss at u is redefined as,

$$V(u) \equiv \inf_{y \in C(u)} \int_0^\infty (y(t') - t')^2 \lambda e^{-\lambda t'} dt'.$$

Moreover, the principal's problem is equivalent to

$$\inf_{t_1 \geq 0, a_1 \geq 0} \int_0^{t_1} (a_1 - t')^2 \lambda e^{-\lambda t'} dt' + e^{-\lambda t_1} V\left((a_1 - t')^2 + \rho(t_1 - R(t_1)\right). \tag{5}$$

The goal of this section will be to derive properties of the optimal continuing allocations at each u.

[16] These papers optimize over the entire allocation subject to the envelope version of the incentive constraint. They find Lagrange multipliers under which this problem is convex and use first order conditions to find the optimal allocation. In the current problem, each allocation induces reputations which (i) factor into the envelope constraints and (ii) are expectations over the inverse mapping of the allocation and thereby do not change smoothly with the allocation.

[17] The compactness of the type space is used in the existence results in earlier sections. In this section, I will prove that the optimal allocation exists using a separate argument.

4.1. Recursive Formulation

Let y be a continuing allocation at u with first threshold t. This first threshold pins down the reputation for this interval as $R(t)$. Since $u = L^A(0|y) = y(0)^2 - \rho R(t)$, this first threshold also pins down the first action $y(0)$. Finally, $y(0)$ and t together pin down the loss at the first threshold $L^A(t|y)$. Motivated by this, define

$$a(t,u) \equiv \sqrt{u + \rho R(t)} \text{ and}$$
$$\overline{u}(t,u) \equiv (a(t,u) - t)^2 + \rho(t - R(t)).$$

The minimized continuation losses must choose this first threshold optimally.[18]

$$V(u) = \inf_{t \geq 0} \int_0^t \lambda(a(t,u) - t')^2 e^{-\lambda t'} dt' + e^{-\lambda t} V(\overline{u}(t,u)) \quad \forall u. \tag{6}$$

This is a Bellman equation with both a one dimensional state variable – u, and a one dimensional control variable – t. The natural question is whether the converse to the above implication holds: if a set of continuation losses satisfy (6), then are these continuation losses minimized? The next result answers a stronger version of this question affirmatively.

Proposition 2. *Take $\underline{u} \leq \rho^2/4$ and $\{y_u\}$ be a set of continuing allocations at each $u \geq \underline{u}$. Suppose that $\forall u' \geq \underline{u}$ $L^P(y_u)$ is differentiable in u and $\frac{dL^P(y_u)}{du} \leq 1$.*

$$L^P(y_u) = \min_{t \geq 0} \int_0^t \lambda(a(t,u) - t')^2 e^{-\lambda t'} dt' + e^{-\lambda t} L^P(y_{\overline{u}(t,u)}) \quad \forall u \geq \underline{u} \tag{7}$$
$$\implies L^P(y_u) = V(u) \quad \forall u \geq \underline{u}.$$

It is worth noting a few caveats to Proposition 2. First and most salient, one only needs to verify the recursive condition for $u \geq \underline{u}$. This is due to the following regularity property of the exponential model.

Lemma 5. *In the exponential model, if $u \leq \rho^2/4$ then $\forall t > 0,\ \overline{u}(t,u) > u$, and if $u \geq \rho^2/4$ then $\forall t > 0,\ \overline{u}(t,u) > \rho^2/4$.*

That is, if initial loss $u \leq \rho^2/4$, then no subsequent threshold can admit a loss less than u. This means that the continuing allocations at losses less than u do not have any impact

[18] The objective takes the form of an infimum instead of a minimum to allow for the allocation which pools all types, i.e. taking the first threshold $t \to \infty$.

on the choice of continuing allocations at losses greater than u. This simplification will be especially useful in the next section.

The second caveat is that, unlike in analogous dynamic optimization results, one cannot "ignore" the continuing allocations, and simply check that an arbitrary function $\tilde{V}(u)$ satisfies (7). For example, consider $\tilde{V}(u) = 0\ \forall u \geq 0$. This satisfies (7) by simply choosing $t = 0$, but 0 is clearly not an attainable continuation loss for any u. For this reason, Proposition 2 requires that the continuation losses correspond to actual continuing allocations.[19]

Finally, the continuation losses are required to have a derivative less than 1. This condition guarantees an approximation result in the proof. One may be concerned that this condition makes the result vacuous, but any set of optimal continuing allocations must also satisfy this condition. To see this, let y_u be a continuing allocation at u. Now consider constructing another continuing allocation at $u + \varepsilon$ by maintaining the same set of thresholds as y_u. This means that the change in loss is only the effect of changing the action on each interval to match an increase in the constraint in the initial loss. One can show that the change in loss degrades in the type, i.e. for two types $t'' > t'$, $\frac{d\,L^A(t'|y_u)}{d\,u} > \frac{d\,L^A(t''|y_u)}{d\,u}$. Since $\frac{d\,L^A(0|y_u)}{d\,u} = 1$, by the definition of a continuing allocation, the expectation over these changes is less than 1. Clearly an optimal set of continuing allocations would change the thresholds with u to further decrease the principal's loss,[20] and so this derivative assumption does not restrict the set of allocations.

Proposition 2 provides for the "guess and check" method to solve for the optimal continuation loss. One can conjecture a set of continuing allocations and check that its associated continuation losses satisfy (6). This is the approach taken in the next subsection.

4.2. Separating in the Exponential Model

A focal continuing allocation at u is the separating continuing allocation d_u introduced in the previous section.

Theorem 2. *In the exponential model, $u \geq \rho^2/16 \implies V(u) = L^P(d_u)$.*

The proof of Theorem 2 shows that for $u \geq \rho^2/16$, setting $y_u = d_u$ satisfies (7). While the result says that there exists initial losses such that separating is optimal, it does not speak to whether separating will actually arise in the optimal allocation. However, using logic similar to that behind Lemma 5, one can deduce that loss increases throughout the

[19] In general choosing $t = 0$, will always guarantee that the RHS of (6) is weakly greater than the LHS. This means that loss function iteration can only work to revise the losses downward.

[20] The argument is actually exact under an optimal set of continuing allocations because any change in the thresholds has a 0 effect on $V(u)$ by an envelope theorem argument.

type space until $u \geq \rho^2/4$. This means that, as long as there is an unbounded sequence of thresholds in the optimal allocation (i.e. there is no eventual pooling), then the optimal allocation will eventually be separating. This can be shown analytically for small relative biases.[21]

Lemma 6. *If $\lambda\rho \leq 4$, the pooling continuing allocation at u given by $x(t) = \sqrt{u + \rho/\lambda} \equiv p_u \ \forall t \in T$ is not optimal $\forall u$.*

Corollary 1. *An optimal allocation exists. If $\lambda\rho \leq 4$, then there exists $\bar{t}$ such that for any optimal allocation x^*, $\exists \bar{u}$ with $x^*(t) = d_u(t - \bar{t}) + \bar{t} \ \ \forall t \geq \bar{t}$.*

Corollary 1 first restates the existence of an optimal allocation. Because the type space is now unbounded, one cannot immediately apply the methods to prove existence from the previous sections. I first establish the fact that the allocation is separating or completely pooling after some $\bar{t}$. Then standard methods imply an optimal allocation on the compact space $[0, \bar{t}]$ given the separating or pooling continuation loss above $\bar{t}$.

While the argument for Theorem 2 is complicated, a broad intuition is as follows. There are two main reasons why a floor is optimal and in particular does better than the lowest separating allocation d_0. First, the first pooling action does not need to respect any incentive constraint to the left and can be set freely, whereas $d_0(t)$ is constructed to equalize the incentive to obtain a higher reputation with the cost of increasing material loss. Second, the separating action $d_u(t)$ increases infinitely quickly for low types.

Neither of these reasons are present when determining the continuing allocation for large initial losses. First, any continuing allocation at u must respect the left incentive constraint that the initial loss is u. Second, the loss under the separating continuing allocation increases very slowly for large initial losses. As shown in Figure 2, at initial losses close to $\rho^2/4$ the separating loss is near constant in the type. This means that the comparison between any continuing allocation which pools some first set of types and the separating allocation is more favorable to the latter at large initial losses.

4.3. The Uniform Limit

The separating continuing allocation is not optimal for small initial losses. More generally, it is difficult to analytically solve for the optimal continuing allocation at every u. This is partly because there is no explicit representation of the separating continuation loss,

[21] This is an analytical limitation rather than reflecting that completely pooling is optimal for some parameters. The numerical methods in Subsection 4.4 never reveal completely pooling as optimal.

which by Theorem 2 will be optimal for large u. However, finding $V(u)\ \forall u$ is not necessary to solve for the optimal allocation. If one can instead show that the floor (t_0, a_0) in (4), must be optimally set so that the loss at the first threshold – $(a_0 - t_0)^2 + \rho(t_0 - R(t_0))$– is greater than $\rho^2/16$, then the optimal continuing allocation is separating by Theorem 2. This requires bounding $V'(u)$ for small initial losses. This is tractable in the uniform limit, i.e. when $\lambda \to 0$. The result, reported below, is that the optimal allocation uses a floor and then separates thereafter. The optimal allocation in the uniform limit is illustrated in Figure 5.

Proposition 3. *There exists $c > 0$ such that for $\lambda \leq c$, the optimal allocation is given by*

$$x^*(t) \equiv \begin{cases} a_0 & t < t_0 \\ d_{\tilde{u}}(t - t_0) + t_0 & t \geq t_0 \end{cases},$$

where $\tilde{u} \equiv (a_0 - t_0)^2 + \rho(t_0 - R(t_0)) > \rho^2/16$, and a_0, t_0 solve,

$$\min_{a_0, t_0} \int_0^{t_0} (a_0 - t')^2 \lambda e^{-\lambda t'} dt' + L^P(d_{\tilde{u}}) e^{-\lambda t_0}.$$

In addition,

$$\lim_{\lambda \to 0} a_0 \to k_1 \rho$$
$$\lim_{\lambda \to 0} t_0 \to k_2 \rho$$

where $k_1 < k_2$ are constants.[22]

4.4. Numerical Solution

Getting an explicit solution for the exponential model is difficult because the separating continuing allocation and thereby the separating continuation loss do not have explicit representations. The previous section shows that the solution admits a single pooling interval and then fully separates thereafter when λ is small. While the optimization in (6) can be analytically difficult, it turns out to be numerically simple. This section numerically solves for the optimal allocation for certain parameterizations and shows that these optima mirror the behavior of the uniform limit.

First I make the observation that the problem can be reduced to one with a single parameter. One can show that the minimized continuation loss $V(u)$ in a problem with parameters ρ and λ is the same as that for one with parameters $\rho' = 1$ and $\lambda' = \lambda\rho$ multiplied

[22] Specifically, $k_1 \equiv \frac{\sqrt{17}-1}{16}$, $k_2 \equiv \frac{5\sqrt{17}-13}{32}$, and $k_3 \equiv (k_2 - k_1)^2 + k_2/2$.

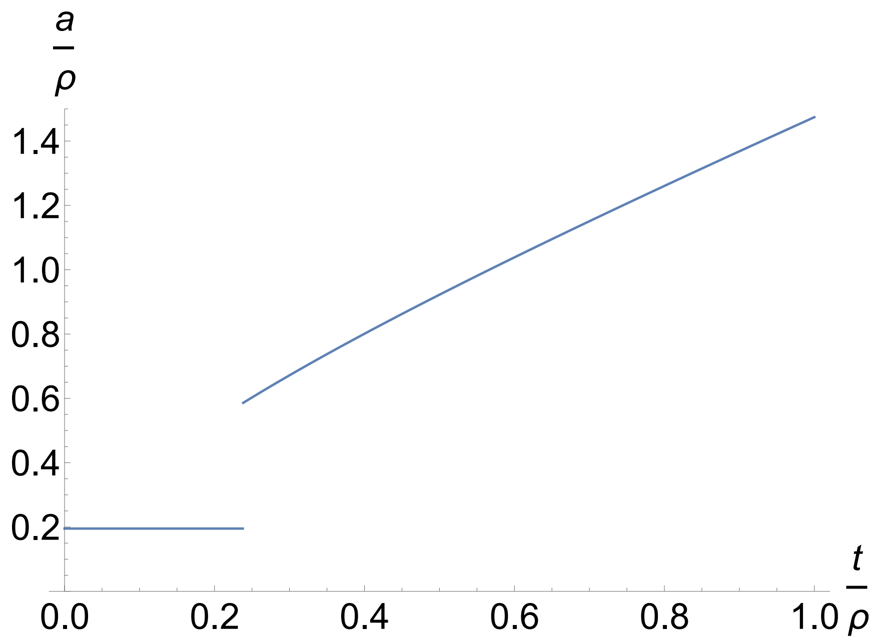


Figure 5: Optimal Allocation in the Uniform Limit.

by the constant ρ^2.[23] This simplification is also represented in the uniform limit solution as the optimal first action and threshold are linear in ρ. With this simplification, one can solve for the optimal allocation for the class of problems in which $\lambda\rho$ is a constant.

The key simplicity is that solving for the minimized continuation loss recursively happens in only one step. The first step starts from the separating continuing allocation d_u as a conjecture, i.e. $\forall u \geq 0$, $y_{0,u}(t) = d_u(t)$ and $V_0(u) \equiv L^P(d_u)$. For each $u \geq 0$, one can solve for the ith iteration by finding

$$t_i^*(u) = \text{argmin}_{t\geq 0} \int_0^t \lambda(a(t,u) - t')^2 e^{-\lambda t'} dt' + e^{-\lambda t} V_i(\overline{u}(t,u)) \quad \forall u,$$

[23] To see this, normalize $\tilde{t} \equiv t/\rho$, $\tilde{u} \equiv u/\rho^2$, and $\tilde{\lambda} \equiv \lambda\rho$. Now define $\tilde{R}(\tilde{t}) \equiv \mathbb{E}[t'|t' \leq \tilde{t},\ t' \sim \tilde{\lambda} e^{-\tilde{\lambda} t'}]$, $\tilde{a}(\tilde{t},\tilde{u}) = \sqrt{u + \tilde{R}(\tilde{t})}$, and $\tilde{u}(\tilde{t},\tilde{u}) \equiv (\tilde{a}(\tilde{t},\tilde{u}) - \tilde{t})^2 + (t - \tilde{R}(\tilde{t}))$.

$$V(u) = \inf_{t\geq 0} \int_0^t (a(t,u) - t')^2 \lambda e^{-\lambda t'} dt' + e^{-\lambda t} V(\overline{u}(t,u))$$
$$\iff V(u) = \rho^2 \inf_{\tilde{t}\geq 0} \left(\int_0^t (\tilde{a}(\tilde{t},\tilde{u}) - t')^2 \tilde{\lambda} e^{-\tilde{\lambda} t'} dt' + e^{-\tilde{\lambda} t} V(\tilde{u}(\tilde{t},\tilde{u})) \right).$$

and then setting the continuing allocation

$$y_{i,u}(t) \equiv \begin{cases} a(t_i^*(u), u) & t < t_i^*(u) \\ y_{i-1,\overline{u}(t_i^*(u),u)}(t - t_i^*(u)) & t \geq t_i^*(u) \end{cases},$$

and the continuation loss $V_i(u) = L^P(y_{i,u})$.[24]

Numerically, it turns out that $V_i(\cdot) = V_{i+1}(\cdot)\ \forall i \geq 1$. By Proposition 2, this means that $y_{1,u}$ is an optimal continuing allocation for every u. Figure 6 displays this recursive structure for every $i \geq 1$. First note that by Theorem 2, $t_i^*(u) = 0\ \forall u \geq 1/16,\ \forall i \geq 0$, i.e. the separating continuing allocation is optimal. Indeed, this is the limit of the region where the separating continuing allocation is optimal as $\lambda\rho \to \infty$. This region tends to become larger as $\lambda\rho$ decreases.[25]

The important feature of Figure 6 is that the optimal first threshold always induces a loss at which separating is optimal. That is, the continuation loss relevant for the first threshold choice is always the first separating conjecture $V_0(u) = L^P(d_u)$.

Remark 1. The convergence of the minimized continuation loss after one step is driven by intuitive (and numerically "verifiable") properties of the problem that are again difficult to demonstrate analytically. Roughly the idea is to formulate the problem in (6) as choosing a next threshold state variable $\tilde{u} \geq u$ instead of choosing the first threshold itself, i.e.

$$V_i(u) = \inf_{\tilde{u} \geq u} \tilde{V}_i(u, \tilde{u}) \equiv \int_0^{\tilde{t}} (a(\tilde{t}, u) - t')^2 \lambda e^{-\lambda t'} dt' + e^{-\lambda \tilde{t}} V_i(\tilde{u}),$$

where $\tilde{t}$ is uniquely defined as $\overline{u}(\tilde{t}, u) = \tilde{u}$ for $u \leq 1/16$. It seems that $\tilde{V}_i(u, \tilde{u})$ has the following single crossing property for $i = 0$ and $i = 1$: $\forall \tilde{u}_1 < \tilde{u}_2,\ \forall u' \geq u$,

$$\tilde{V}(u, \tilde{u}_2) - \tilde{V}(u, \tilde{u}_1) \geq 0 \implies \tilde{V}(u', \tilde{u}_2) - \tilde{V}(u', \tilde{u}_1) > 0.$$

This property gives that the optimal next threshold loss $\tilde{u}^*(u)$ is decreasing in u if the constraint that $\tilde{u} \geq u$ does not bind. Now consider the lowest initial loss u such that separating is optimal $\forall u' \geq u$. Such a u is guaranteed by Theorem 2. Because the RHS of (6) is continuously differentiable, it must be that the constraint that $\tilde{u} \geq u$ does not bind at u. This means that with the aforementioned single crossing property, $\tilde{u}^*(u') \geq u\ \forall u' \leq u$. That is,

[24] If there are multiple solutions, take $t_i^*(u)$ to be the maximum solution. This is important because if $V_i(u) = V_{i+1}(u)$, then $t = 0$ is always a solution at the $i + 1$th iteration. If the Bellman is minimized as $t \to \infty$ set $t_i^*(u) = \infty$.

[25] As proved in the appendix, as $\lambda\rho \to 0$, the region where separation is optimal converges to $u \geq \rho^2/36$.

the separating continuing allocation is optimal at every $\tilde{u}^*(u)$, and $V_i = V_1 \; \forall i \geq 1$.

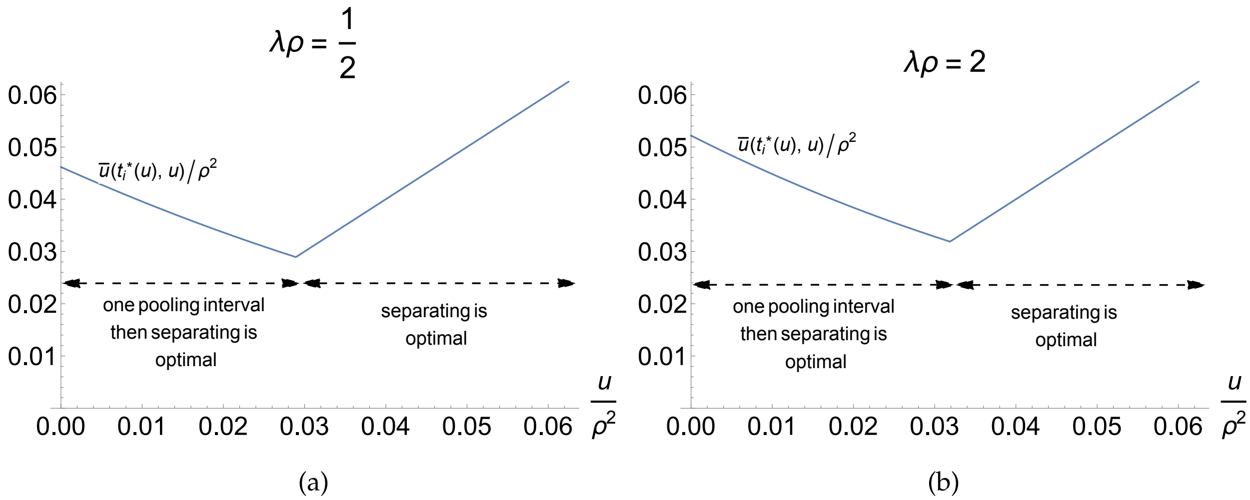


Figure 6: The agent's loss at the optimal first threshold.

As mentioned earlier, solving for the optimal continuing allocations at each u is one step away from solving for the optimal allocation. In particular given $V_1(u) = V(u)$, one needs to solve for the first threshold and action a_0, t_0 in (5). In theory, this could mean that even though each optimal continuing allocation has at most one pooling interval, the optimal allocation could have two pooling intervals. However, this does not turn out to be the case. Figure 7 shows the optimal allocations for two parameter specifications below. The optimal allocation has one pooling interval in each case, that is, a floor and then separating is optimal just like in the uniform limit.

The fact that there is one pooling interval in the optimal allocation means that for the optimal choice of a_0, t_0 in (5) means that the separating continuing allocation is optimal at the first threshold loss, i.e. $V(u_0) = L^P(d_{u_0})$ where $u_0 \equiv (a_0 - t_0)^2 + \rho(t_0 - R(t_0))$. Indeed, one can show that at the optimum $u_0 > \rho^2/16 \; \forall \lambda, \rho > 0$, so this conclusion is delivered by Theorem 2. One reason for this is that the incentive to increase the first threshold loss is greater in (5) as compared to that in the continuing allocation problem for any u. This in turn is due to the fact that in (5) the principal sets a_0 optimally as opposed to it being determined exogenously in (6) as $a(t, u)$. The increase in $a(t, u)$ when the first threshold t increases incurs an extra source of loss for the principal in the continuing allocation problem relative to the determination of the first threshold in (5).

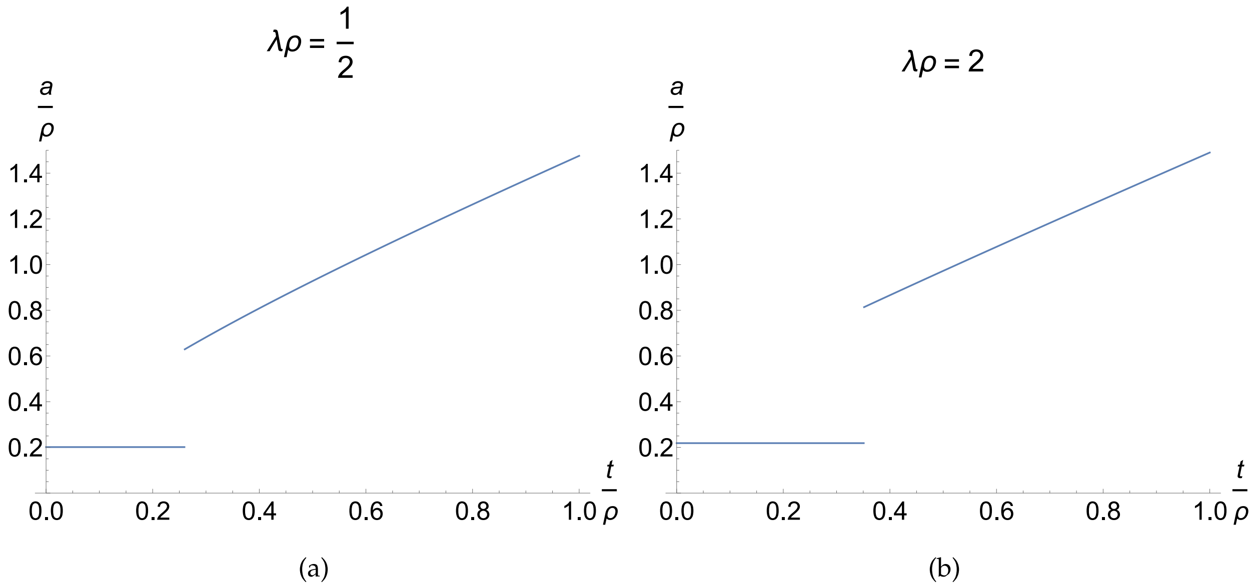


Figure 7: The Optimal Allocation.

5. Unique Implementation

For expositional purposes, the paper focuses on the allocation $x : T \to A$. This ignores questions of equilibrium multiplicity, and how strongly the principal can implement his preferred allocation through delegation. In this subsection I show that, under a regularity condition on the distribution of types, the principal can implement (almost) any IC allocation as the unique equilibrium that satisfies the D1 criterion introduced by Cho and Kreps (1987).

Given a delegation set $\tilde{A}$ and an intended IC allocation $x : T \to \tilde{A}$. There are two sources of equilibrium multiplicity that may cause concern: (i) there may exist alternative incentive compatible allocations that induce the same set of chosen actions but allocate them to different types, and/or (ii) there may exist incentive compatible allocations which induce a strict subset of $\tilde{A}$. In order to address the second issue, one must explicitly consider off path beliefs. This consideration was irrelevant in the main text analysis as only equilibria which used all delegated actions were considered.

An allocation is now a pair $(x, \tilde{A})$ where $\tilde{A} \subset A$, and $x : T \to \tilde{A}$. An incentive compatible allocation is $(x, \tilde{A})$ such that there exists $\mu : \tilde{A} \setminus x(T) \to \Delta T$ such that $\forall t \in T$,

$$L^A(t|x) = \min_{t' \in T} L^A(x(t'), t|x), \text{ and}$$
$$L^A(t|x) \geq (a-t)^2 + \rho(t - \mathbb{E}[t'|t' \sim \mu_a]) \quad \forall a \in \tilde{A} \setminus x(T).$$

This definition puts no structure on the off-path beliefs, and as a result many incentive compatible allocations $(x, \tilde{A})$ with $x(T) \subsetneq \tilde{A}$ may exist. However, these equilibria make use of punitive off-path beliefs, e.g. assuming that any unchosen action is taken by type $t = 0$. I focus instead on beliefs that satisfy the D1 refinement which has proven useful and reasonable in signaling games like that induced between the agent and the market. Consider an allocation $(x, \tilde{A})$. In this context, the D1 refinement says that given $a \in \tilde{A} \backslash x(T)$, if $\exists t', t''$ such that

$$\begin{aligned} &\{\tilde{R} \in [0, M] : \; L^A(t'|x) \leq (a - t')^2 + \rho(t' - \tilde{R})\} \\ \subsetneq &\{\tilde{R} \in [0, M] : \; L^A(t''|x) < (a - t'')^2 + \rho(t'' - \tilde{R})\}, \end{aligned} \tag{8}$$

then the off path belief has $t' \notin \text{Supp}(\mu(a))$. That is, if deviation to an off-path action is tempting for a strictly larger set of off-path beliefs for one type than another, then the market should weight the former type arbitrarily highly relative to the latter type.

In order to deal with equilibrium multiplicity over allocations that use the same set of actions, I show that a regularity condition on the distribution of types rules out this kind of non-uniqueness.

Definition 1. The distribution satisfies condition (M^*) if $\forall a_1 < a_2$, and $\forall t_1 < t_2$ the expression

$$\big((a_2 - t)^2 - \rho R(t, t_2)\big) - \big((a_1 - t)^2 - \rho R(t_1, t)\big) \tag{9}$$

is single crossing from above in $t \in [t_1, t_2]$.[26,27]

To interpret condition (M^*) consider that an allocation splits an interval of types $[t_1, t_2]$ by assigning $[t_1, t)$ to action a_1 and $[t, t_2]$ to action a_2. Condition (M^*) says that moving the threshold type t rightward can never make that type switch from preferring the high action to preferring the low action. From a material loss standpoint, moving the threshold type t rightward results in a stronger preference for the higher action a_2. Thus condition (M^*) imposes that the changes in reputational differences never overwhelm this material change. A sufficient condition for condition (M^*) is that the density is weakly decreasing so that $R(t, t_2) - R(t_1, t)$ is increasing in t. Thus two examples that satisfy the condition are the uniform distribution and the exponential distribution. However, the condition also permits distributions such that the $R(t, t_2) - R(t_1, t)$ does not decrease too fast relative to the change in material loss.

[26] A function $g : T \to \mathbb{R}$ is single crossing from above if $\forall t_1 < t_2 \; g(t_1) \leq 0 \implies g(t_2) < 0$.

[27] If $T = [0, \infty)$ is infinite, I also impose the condition to hold for "$t_2 = \infty$".

Lastly, say that $\tilde{A} \subset A$ is a **finite delegation set** if one can write $\tilde{A} = A_1 \cup A_2 \cup ... \cup A_n$, where each $A_i \subset A$ is an (potentially degenerate) interval. Note that restricting to finite delegation sets does not mean delegating a finite set of actions – finite delegation sets can have an arbitrary number of separating regions, i.e. when A_i is a non-degenerate interval.

Proposition 4. *Let the distribution of types satisfy condition (M^*). Let $\tilde{A} \subset A$ be a finite delegation set and $x : T \to \tilde{A}$ be an incentive compatible allocation such that $x(T) = \tilde{A}$. If $(y, \tilde{A})$ is incentive compatible and satisfies the D1 refinement then $y = x$.*

The result says that delegating the range $x(T)$ of an incentive compatible allocation implements this allocation as the unique D1 equilibrium between the agent and the market. The D1 refinement puts a lot of structure on off-path beliefs in this model. In particular, if an action a is off-path under $(y, \tilde{A})$, then the reputation for choosing action a puts probability 1 on $t = \min\{y^{-1}(a') : \ a' > a\}$, i.e. the lowest type who takes an action higher than a. This combined with condition (M^*) gives uniqueness.[28]

To illustrate the result, consider the optimal allocation in the exponential model in the uniform limit. This is spelled out by Proposition 3 as

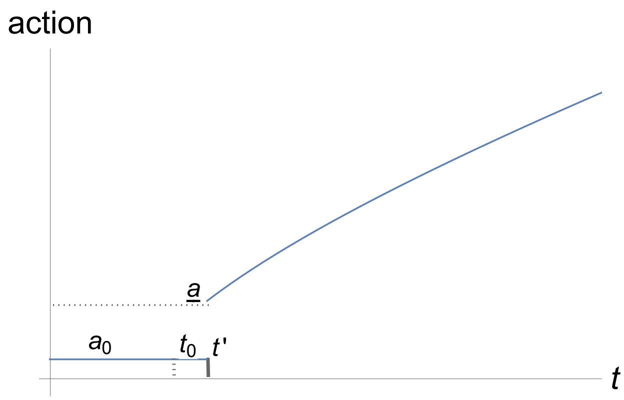
$$x^*(t) = \begin{cases} a_0 & t < t_0 \\ d_{(\underline{a}-t_0)^2}(t - t_0) & t \geq t_0 \end{cases},$$

where $(\underline{a} - t_0)^2 = (a_0 - t_0)^2 + \rho(t_0 - R(t_0))$. This means that the principal can implement this allocation with the finite delegation set $\tilde{A} = \{a_0\} \cup [\underline{a}, \infty)$. One may be concerned about whether there are other incentive compatible allocations $y : T \to \tilde{A}$ such that $y \neq x$. Proposition 4 and the fact that the exponential distribution satisfies condition (M^*) says that there is no such allocation y that satisfies the D1 refinement. As mentioned there are two kinds of non-uniqueness to rule out. Figure 8 below illustrates two associated alternative allocations and why they are not D1 incentive compatible. The left panel displays an alternative allocation that uses all actions, i.e. $y(T) = \{a_0\} \cup [\underline{a}, \infty)$. Whether or not this is incentive compatible comes down to whether there exists another solution besides t_0 to the equation

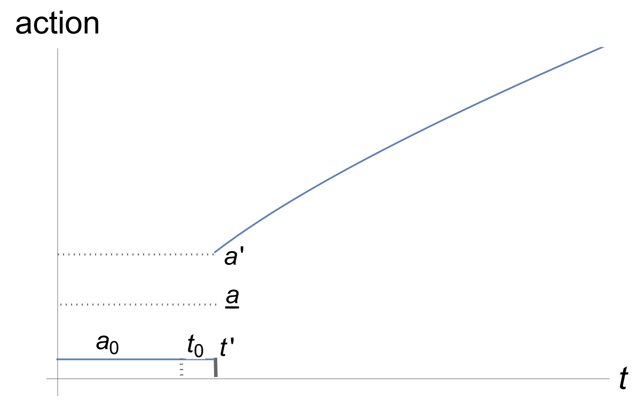
$$(\underline{a} - t')^2 - (a_0 - t')^2 = \rho(t' - R(t')).$$

[28] The role condition (M^*) plays in guaranteeing uniqueness is related to that for condition (M) from Crawford and Sobel (1982) (this motivates the name). When $\tilde{A} = \{a_1, a_2, ...a_n\}$ has a finite set of actions, an incentive compatible allocation can be constructed via a difference equation that pins down the "next" threshold as a function of the last two. As pointed out in Crawford and Sobel (1982), this is also true in the standard cheap talk model, and condition (M) there guarantees that there is a unique starting threshold such that the final threshold is the supremum of the type space. The proof of Proposition 4 uses condition (M^*) similarly.

However, notice that the right hand side is increasing in t' for the exponential distribution,[29] and the left hand side is decreasing in t', because higher types have relatively lower loss from higher actions. Thus there is at most one solution to this equation. This implies that if we consider an alternative first threshold $t' > t_0$, then t' will strictly prefer $\underline{a}$ over a_0. One potential solution to restore incentive compatibility would be to assign a higher action $a' > \underline{a}$ to type $t' > t_0$, and start the separating continuing allocation from there. The right panel displays such an allocation which uses a strict subset of the actions. However, in this case $\underline{a}$ is off-path and under the D1 refinement commands a reputation of t'. Thus, $\underline{a}$ remains a profitable deviation for t'.



(a) t' prefers $\underline{a}$ to a_0.



(b) t' prefers $\underline{a}$ to a'.

Figure 8: Alternative allocations with the same delegation set.

Remark 2. A weaker version of condition (M^*) is to impose that the expression in (9) is single crossing only for actions in the given delegation set $\tilde{A}$. The proof of Proposition 4 then directly implies uniqueness among all D1 IC allocations $x : T \to \tilde{A}$, and that the unique such allocation uses all actions, i.e. $x(T) = \tilde{A}$. This is especially useful when T is infinite but one knows that the optimal allocation is eventually pooling or separating after some fixed type $\bar{t}$. Let $A_n = [\underline{a}, \infty)$, and $y : T \to \tilde{A}$ be a D1 IC allocation, then $A_n \subset y(T)$.[30] This will imply that one only needs to impose condition (M^*) for a compact set of actions in order to get unique implementation of the optimal allocation.

[29] This property holds for any log-concave distribution.

[30] This follows from Claim 15, the argument for which says that $y(T) \cap A_n$ is either a singleton, or it includes $[\underline{a}, \tilde{a})$ for some $\tilde{a} > \underline{a}$, and then possibly includes $\max\{A_n\}$. In the case where T is infinite and $A_n = [\underline{a}, \infty)$, there is no maximum action and the rest of the cases are ruled out by the following argument. If $a = \max\ A_n \cap y(T)$, $a + \varepsilon$ has lower material loss than a for high enough types and therefore has infinite reputation under the D1 refinement.

6. Discussion

6.1. Material Delegation vs. Reputational Delegation

A floor being optimal when delegating to an agent with a reputational bias is in stark contrast to optimal delegation to an agent with a material bias. For example consider the canonical material delegation model in which the principal's and agent's loss functions given allocation x are given by,

$$L^A(a,t|x) = (a-(t+b))^2$$
$$L^P(x) = \int_0^M (a-t)^2 f(t)dt$$

with $b > 0$. This is a material delegation model because the agent's loss does not depend on the allocation beyond his type and choice of action. These specific preferences are not arbitrary: the cheap talk game between the principal and the agent, i.e. one in which the principal lacks commitment, has the same equilibrium set as that for the preferences in the main text reputational model if $\rho = 2b$.[31] Alonso and Matouschek (2008) derive that the optimal delegation set for these preferences and a set of "regular" distributions is to fully delegate up to some cap C, i.e. to delegate the set of actions $[0, C]$.

Even if the preferences and distribution do not entail interval delegation to an agent with a material bias, it is never optimal to restrict flexibility for low types as long as the agent is upward biased. To make this point more precise, consider any continuous loss functions for the agent and principal, $l^A : A \times T \to \mathbb{R}$ and $l^P : A \times T \to \mathbb{R}$ that are both strictly submodular in (a,t), strictly convex in a, $\forall t$, and such that $\forall t \in T$, and $\forall a_1 < a_2 \in A$,

$$l^A(a_1,t) \leq l^A(a_2,t) \implies l^P(a_1,t) < l^P(a_2,t).$$

The displayed condition says that the principal always strictly prefers a lower action if the agent weakly prefers it, i.e. the agent is upward biased relative to the principal. Now let

[31] A cheap talk equilibrium in both environments is characterized by its set of thresholds. It is necessary and sufficient for a set of thresholds to constitute a cheap talk equilibrium if each threshold is indifferent between the actions taken for types to the left and right. The action for any message sent by an interval of types $[t_1, t_2]$ is pinned down by receiver optimality as the expectation, i.e. $R(t_1, t_2)$. Let $t_1 < t_2 < t_3$ be three sequential thresholds:

$$(R(t_1,t_2)-t_2)^2 + \rho(t_2 - R(t_1,t_2)) = (R(t_2,t_3)-t_2)^2 + \rho(t_2 - R(t_2,t_3))$$
$$\iff (R(t_1,t_2)-(t_2+\rho/2))^2 = (R(t_2,t_3)-(t_2+\rho/2))^2.$$

That is, indifference at the thresholds is the same condition for the agent with a reputational bias ρ as for an agent with a material bias $\rho/2$.

$a^*(t) \equiv \arg\min_{a\in A} l^A(a,t)$. For any density f, every optimal allocation has $x(0) \leq a^*(0)$.[32] In addition, it is without loss of optimality to delegate $[0, x(0)) \cup x(T)$, i.e. full flexibility at the bottom is optimal. This means that we cannot find a material delegation model with a purely upward material bias that has the same optimal allocation as the reputational delegation model.

6.2. Alternative Material Preferences

The main analysis is carried out with the assumption of quadratic loss as the form of material preferences of both the agent and the principal. It can be shown that, for Theorem 1, this analysis extends to any convex continuously differentiable loss function of the distance between the chosen action and the type, i.e. $l(|a-t|)$. The key points used in the analysis are that, (i) $l'(0) = 0$, and that (ii) l is shared as the material loss for both the principal and the agent.

Point (i) guarantees that in order to incentivize the agent to separate for low types, the principal must increase the action dramatically in order to deter the agent from seeking a small reputational gain from misreporting. To see this, consider a separating allocation $x : T \to A$ under such a general loss function. According to the analysis in Section 2, it is analogously pinned down by an initial loss condition and the following differential equation,

$$x'(t) = \frac{\rho}{l'(|x(t)-t|)}.$$

Because $l'(\varepsilon) \approx 0$ for small ε, when the agent is experiencing small losses, deterring a small reputational gain requires an arbitrarily large increase in the action.

Point (ii) is sufficient in guaranteeing the alignment principle– Proposition 1– which is key to the intuition of an optimal floor. The alignment principle relies on the fact that the agent is upward biased relative to the principal, i.e. if an agent type is indifferent between two actions in an allocation then the principal will prefer that the lower action be assigned to that type. Given that the agent and principal share the same material preference, the agent preferring to be seen as a higher type, i.e. his reputational bias, is sufficient to induce this kind of upward bias. If the agent's material preferences were negatively biased rela-

[32] To see why, suppose that an optimal allocation has $x(0) > a^*(0)$. Now take the type $\tilde{t}$ to be the maximum type such that $a^*(\tilde{t}) \leq x(0)$. Note that by incentive compatibility $x(t) = x(0)\ \forall t \in [0, \tilde{t}]$. Since the agent has an upward bias, the following incentive compatible allocation improves on x:

$$y(t) \equiv \begin{cases} a^*(t) & t < \tilde{t} \\ x(t) & t \geq \tilde{t} \end{cases}.$$

tive to the principal the agent may no longer have a net upward bias, and the alignment principle may not go through.

The main text analyzes the case of no material bias to obtain the starkest comparison between existing results. However, the above paragraph suggests that the current analysis can deal with misaligned material preferences between the principal and agent if it in fact reinforces the agent's upward bias. For example, if the principal has a loss function $l(|a - (t-b)|)$ with $b > 0$ while the agent has the same preferences as in the main text, then the agent will be upward biased relative to the principal.[33] Under these assumptions the alignment principle goes through unchanged. One can show that a version of Theorem 1 holds in this new environment. In specific, the same arguments imply that there will be a non-trivial first pooling interval at any optimum. However, this interval of types no longer necessarily pools on an action that is above the ideal point of the lowest type agent $t = 0$, i.e. it is no longer a "floor". A simple example is one in which b is very large so that the optimal action for the principal given the highest type is negative. This would in turn imply that pooling every type on this action is the optimal allocation.

The explicit characterization results in Section 4 depend heavily on quadratic loss. However the recursive approach in Proposition 2 would be valid for any symmetric loss functions described above, as long as Lemma 5 holds.

6.3. Alternative Reputational Preferences

The main analysis is carried out with the assumption that reputational vaue for the agent from a given belief $\mu \in \Delta T$ is proportional to $\mathbb{E}[t|t \sim \mu]$. It can be shown that the analysis extends directly to the case in which this value is proportional to $\mathbb{E}[r(t)|t \sim \mu]$ where r is a continuously differentiable function with derivative bounded away from 0.

As mentioned in the previous subsection, one key feature is maintaining the upward biased preferences of the agent relative to the principal. Given any additively separable and type independent reputational value over beliefs, any allocation will induce a submodular agent loss over actions and types. This means that higher actions will be taken by higher intervals of types. Thus the agent is upward biased relative to the principal if the reputational value for the prior conditioned on a higher interval is larger than that for a lower interval. Given this point, It seems like this analysis could also extend to "non-expectational" reputational values of the belief as long as they guarantee this associated monotonicity property.

[33] In this setting the action space would be augmented to $[-b, \infty)$ in order to include both party's optimal actions for every type.

As for the results in Section 4, the exact form of the reputational preference is important. There is only one state variable – the initial loss– in (6) because the reputational loss for a given interval only depends on the interval's length. This memorylessness would not hold for the expectation of a non-linear function of the type.

7. Conclusion

This paper studies whether an agent with a reputational bias should be treated differently than an agent with a material bias. Despite the models having many superficial similarities, I answer this question affirmatively. Specifically, Theorem 1 showed that unlike in material delegation where it is never beneficial to restrict flexibility to low type agents, it is always optimal to impose a floor in reputational delegation. In addition, this is not due to flexibility being sub-optimal per-se. Theorem 2 shows that it is optimal to give full flexibility in the exponential model, when the initial loss is large.

To solve the exponential model, I use the recursive nature of the problem, namely that the first threshold in any continuing allocation pins down the initial loss at that threshold, and thereby the minimized continuation loss for the principal. This method is not specific to the reputational delegation framework, and can be used in other difficult mechanism design problems. The important features are that the allocation over which the principal has commitment (the action) is monotone in the type, and the portion over which the principal does not have commitment (the reputation) is dependent only on the chosen action.

Finally, I assumed throughout the paper that the agent benefits from having a reputation of being a higher type. As discussed, this assumption fits many principal-agent interactions, however it was also partially chosen to make the model directly comparable with the standard material delegation framework in which the agent is biased towards higher actions. It would be interesting for future work to study optimal delegation to agents with different reputational biases. In particular, standard models of expertise, e.g. Ottaviani and Sorensen (2006a), would associate extreme actions with better agents rather than higher actions. Conversely, Bernheim (1994) studies agents with a "preference for conformity", and this reputational bias would push agent's to take more middling actions.

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A. Preliminaries

I first list and prove a series of lemmas that will be used in the arguments for the main text results. Lemma 7 shows that it is without loss to restrict to a compact action set. Lemma 8 derives the familiar envelope condition version of incentive compatibility. Lemma 9 shows that the minimized continuation loss is differentiable. Lemma 10 gives a representation of the derivative of the agents loss given a change in the initial loss. Lemma 11 shows that the pooling action for a given interval is the minimum action for the endpoint type among all incentive compatible allocations. Lemma 12 uses the previous lemma to derive a lower bound on the loss of the endpoint type for any small interval.

Lemma 13 provides a necessary condition for any optimal pooled interval derived by separating a small set of types near the left endpoint.

For any allocation $x \in IC(T)$ define the endpoint functions $\underline{\tau}, \overline{\tau} : T \to T$ such that $r_x(x(t)) = R(\underline{\tau}(t), \overline{\tau}(t))\ \forall t$. When there is ambiguity, I will notate the allocation in the endpoint functions as $\underline{\tau}_x,\ \overline{\tau}_x$.

It will also be useful to define a sense of a "small interval", such that if this interval is pooled, then the corresponding action is greater than the endpoint type: let $\overline{\delta} > 0$ be defined by $\sqrt{\rho(R(t, t+\delta) - t)} > \delta\ \forall t \in T,\ \forall \delta < \overline{\delta}$. This implies that for $t :\ \overline{t}(t) - \underline{t}(t) < \overline{\delta}$, it holds that $x(t) > \overline{t}(t)$. Since f is bounded away from zero, so is the derivative of $R(t_1, t_2)$ with respect to t_1 and t_2, and so such a $\overline{\delta}$ exists.

Lemma 7. *Let $x \in IC(T)$. There exists an allocation $\tilde{x} : T \to [0, B]$ where $B \equiv \sqrt{\rho M + M^2} + M$ such that $\tilde{x} \in IC(T)$ and $L^P(\tilde{x}) \leq L^P(x)$.*

Proof. Note that if $x(0) \geq M$, then by monotonicity $x(t) \geq M\ \forall t \in T$. This means the principal can improve on x with the allocation $\tilde{x}(t) = R(0, M)\ \forall t \in T$. Thus $x(0) < M$ in any optimal allocation. By incentive compatibility,

$$\begin{aligned}
&(x(M) - M)^2 - (x(0) - M)^2 \leq \rho(r_x^*(M) - r_x^*(0)) \\
\implies &x(M) \leq \sqrt{\rho M + M^2} + M
\end{aligned}$$

Q.E.D.

Lemma 8. *$x \in IC(T)$ if and only if x is increasing and*

$$L^A(t|x) = \int_0^t (\rho - 2(x(t') - t'))\, dt' + L^A(0|x)\ \forall t \in S.$$

Moreover, $L^A(t|x)$ has left and right derivatives given by $\rho - 2(x(t^{+(-)}) - t)$ respectively.

Proof. $\forall a \in [0, B]$ from Lemma 7, and $\forall t \in S\ L_t^A(a, t|x) = \rho - 2(a - t)$ is equicontinuous in t for all $a \in x(S)$. The forward implication thereby follows from theorem 3 in Milgrom and Segal (2002).

To see the reverse implication, take $t_0 < t_1$. The assumption of the lemma gives that

$$L^A(t_1|x) - L^A(t_0|x) = \int_{t_0}^{t_1} (\rho - 2(x(t') - t'))\, dt'$$
$$\iff (x(t_1) - x(t_0))(x(t_1) + x(t_0) - 2t_0) - \rho(r_x^*(t_1) - r_x^*(t_0)) + 2\int_{t_0}^{t_1} (x(t') - x(t_1))\, dt' = 0$$
$$\implies (x(t_1) - x(t_0))(x(t_1) + x(t_0) - 2t_0) - \rho(r_x^*(t_1) - r_x^*(t_0)) \geq 0.$$

The last implication is due to x being increasing. The last line is equivalent to t_0 preferring $x(t_0)$ to $x(t_1)$. The argument for $t_1 < t_0$ is symmetric. *Q.E.D.*

Lemma 9. *$\forall t, u$ $V(t,u)$ has left and right derivatives in u denoted $V_{u^-}(t,u)$ and $V_{u^+}(t,u)$ respectively.*

Proof. In order to show differentiability of V, I will redefine the set of continuing allocations by their induced reputations. Let $\mathcal{R} \equiv \{r_x^*(t) : \ x \in C(t,u)\}$, i.e. the set of reputation functions resulting from any given interval partition of $[t, M]$. First note that for any such reputation function, $r : T \to \mathbb{R}^+$, there exists a continuing allocation $x : T \to A$ at u such that

$$(x(t) - t)^2 + \rho(t - r(t)) = \max_{t' \in T}(x(t') - t)^2 + \rho(t - r(t')).$$

That is, x is incentive compatible given reputations given by r. One can construct such an allocation iteratively.[34] Note that this also implies $\mathcal{R}$ is independent of u.

I will first show that the reputation function uniquely pins down the continuing allocation. Suppose not, i.e. $\exists x, y \in C(t,u)$ such that $r_x^* = r_y^*$ but $x \neq y$. Take $s \equiv \inf\{t' : L^A(t'|x) \neq L^A(t'|y)\}$. Since the agent loss is continuous for any IC allocation and $L^A(t|x) = L^A(t|y) = u$, it must be that $L^A(s|x) = L^A(s|y)$. Suppose first that there exists $t' < s + \overline{\delta}$ such that $L^A(t'|x) \neq L^A(t'|y)$ and $\underline{\tau}(t') < s$. Because the reputations are the same, it must be that $x(t') \neq y(t')$. But since $\underline{\tau}(t') < s$, $x(t') = x(s) \neq y(s) = y(t')$. This means that there exists $\tilde{t} \leq s$ such that $L^A(\tilde{t}|x) \neq L^A(\tilde{t}|y)$, which is a contradiction to the definition of s.[35]

This means that $\forall t' \in [s, s + \overline{\delta}) : \ L^A(t'|x) > L^A(t'|y)$ (a symmetric argument holds for

[34] To see this, suppose an allocation x satisfies the above equation for $t \in [0, \bar{t}]$. Because the reputations correspond to an interval partition it is without loss to take $\bar{t} = \underline{\tau}(\bar{t})$. Now consider the case in which the first threshold $\tilde{t}$ of the interval partition greater than $\bar{t}$ exists and is greater than $\overline{\delta} + \bar{t}$. In this case define $x(t') \equiv \sqrt{L^A(\bar{t}|x) + \rho(R(\bar{t}, \tilde{t}) - \bar{t})}$ $\forall t' \in [\bar{t}, \tilde{t}]$ which is defined because $L^A(\bar{t}|x) \geq 0$. If instead, one cannot find such a threshold, then define $\tilde{t}$ to be the highest threshold in $(\bar{t}, \bar{t} + \overline{\delta}]$. Define $x(t') \equiv \sqrt{l(t') + \rho(r(t') - t')} + t'$, where $l(t)$ solves the differential equation $l'(t) = \rho - 2\sqrt{l(t') - \rho(t - r(t'))}$ on $[\bar{t}, \tilde{t}]$ with initial condition $l(\bar{t}) \equiv L^A(\bar{t}|x)$. A solution exists by theorem 1 in Persson (1975).

[35] $\tilde{t}$ need not be s because it could be that $x(s) - s = s - y(s)$ in which case, despite different actions, $L^A(s|x) = L^A(s|y)$.

the case in which $L^A(t'|x) > L^A(t'|y)$), it holds that $\underline{\tau}(t') > s$. Because agent losses are continuous in the type, there exists $\underline{s} \geq s$ such that $L^A(t''|x) > L^A(t''|y)$ $\forall t'' \in (\underline{s}, t']$ and $L^A(\underline{s}|x) = L^A(\underline{s}|y)$. By definition of $\overline{\delta}$, this means that $x(t'') > y(t'') > t''$ $\forall t'' \in (\underline{s}, t']$. But this is a contradiction, because by Lemma 8,

$$\begin{aligned}&L^A(t'|x) - L^A(t'|y)\\=&2\int_{\underline{s}}^{t'} (y(t'') - x(t''))\, dt'' + L^A(\underline{s}|x) - L^A(\underline{s}|y)\\<&0.\end{aligned}$$

Thus, for each $r \in \mathcal{R}$ define $x_{r,u}$ as the unique continuing allocation at (t, u) with reputation function $r^*_{x_{r,u}}(t') = r(t')$ $\forall t' \in [t, M]$. I can rewrite the problem in (3) as,

$$V(t,u) = \frac{1}{F([t,M])} \min_{r\in\mathcal{R}} \int_t^M L^A(t'|x_{r,u}) f(t')dt'$$

Because $L^A(0|x_{r,u}) = u$ $\forall r \in \mathcal{R}$, $\frac{d}{du}L^A(0|x_{r,u})$ is equicontinuous at $t = 0$. Moreover since, $\rho - 2B < L_t(t'|x_{r,u}) < \rho + 2t'$ $\forall t', r \in \mathcal{R}$, $\frac{d}{du}L^A(t|x_{r,u})$ is equicontinuous at every t in u. Thus the integrand above is equidifferentiable and by theorem 3 in Milgrom and Segal (2002) $V(t, u)$ has left and right derivatives in u.

Q.E.D.

Note that for a given reputation function in $r \in \mathcal{R}$ the set of discontinuities of $x_{r,u}$ is the same as that for r. Define the threshold types of $r \in R$, as $\tilde{J} \subset T^2$ as $\{(\underline{t}, \overline{t}) \in T^2 : \ \underline{t}, \overline{t} \in J, \ (\underline{t}, \overline{t}) \cap J = \emptyset\}$. Define the set of separating and pooling boundaries as $\tilde{J}^s \equiv \{(\underline{t}, \overline{t}) \in \tilde{J} : r(t) = t \ \forall t \in (\underline{t}, \overline{t})\}$ and $\tilde{J}^p \equiv \{(\underline{t}, \overline{t}) \in \tilde{J} : r(t) = R(\underline{t}, \overline{t}) \ \forall t \in (\underline{t}, \overline{t})\}$. Note that by Lemma 1, $\tilde{J}$ is countable and $\tilde{J} = \tilde{J}^p \cup \tilde{J}^s$, and $\tilde{J}^p \cap \tilde{J}^s = \emptyset$.

Lemma 10. *Let J be the set of discontinuities of $x_{r,u}$. $\frac{d\, L^A(t|x_{r,u})}{du}$ is given by the following expression,*

$$\frac{x_{r,u}(t) - t}{x_{r,u}(t) - \underline{\tau}(t)} \left(\prod_{(\underline{t},\overline{t})\in\tilde{J}^p : \overline{t}<t} \frac{x_{r,u}(\underline{t}) - \overline{t}}{x_{r,u}(\underline{t}) - \underline{t}} \right) \left(\prod_{(\underline{t},\overline{t})\in\tilde{J}^s : \underline{t}<t} e^{2/\rho\left((x_{r,u}(\underline{t})-\underline{t})-(x_{r,u}(\max\{\overline{t},t\})-\max\{\overline{t},t\})\right)} \right).$$

Proof. Recall the convention that $x_{r,u}$ is right continuous so for $(\underline{t}, \overline{t}) \in \tilde{J}^p$ $x_{r,u}(\underline{t}) - \underline{t} = \sqrt{L^A(\underline{t}|x_{r,u}) + \rho(R(\underline{t}, \overline{t}) - \underline{t})} > 0$, so the above expression is well defined. Now let $\tilde{t} < M$ be the maximum t' such that the equality holds for $[t, t']$.[36] Suppose first that $\tilde{t} \in (\underline{t}, \overline{t})$ for

[36] Both sides are continuous in t so this is well defined.

some $(\underline{t}, \bar{t}) \in \tilde{J}^p$. Then $x_{r,u}(t') - \underline{t} = \sqrt{L^A(\underline{t}|x_{r,u}) - \rho(\underline{t} - R(\underline{t}, \bar{t}))}$ $\forall t' \in [\underline{t}, \bar{t})$. But by taking derivatives in u, this implies that $\frac{d\, L^A(t|x_{r,u})}{du} = \frac{d\, L^A(\underline{t}|x_{r,u})}{du} \frac{x_{r,u}(t)-t}{x_{r,u}(t)-\underline{t}}$. Since the above equality holds at $\underline{t}$ by assumption, the equality also holds at $\bar{t}$, which contradicts the maximality of $\tilde{t}$. Thus $\underline{\tau}(\tilde{t}) = \tilde{t}$.

Now suppose there exists a first threshold $\bar{t} \in \overline{\tau}([t, M]) \cap (\tilde{t}, M]$. Then $x_{r,u}(t') - \tilde{t} = \sqrt{L^A(\tilde{t}|x_{r,u}) - \rho(\underline{t} - R(\tilde{t}, \bar{t}))}$ $\forall t' \in [\tilde{t}, \bar{t})$, and the argument from the previous paragraph still applies.

The last case is that there exists no first threshold $\bar{t} \in \overline{\tau}([t, M]) \cap (\tilde{t}, M]$. This means that there exists a threshold $\bar{t} \in \overline{\tau}([t, M]) \cap (\tilde{t}, M]$ such that $\bar{t} < \bar{\delta}/2$. This means that the allocation solves the differential equation $L^A_t(t'|x_{r,u}) = \rho - 2\sqrt{L^A(t'|x_{r,u}) + \rho(t' - r(t'))}$, where the positive square root is indicated by $x_{r,u}(t') > t',\ \forall t' \in [0, \bar{t}]$ by the definition of $\bar{\delta}$. Taking the derivative of both sides with respect to u, gives $L^A_{tu}(t'|x_{r,u}) = -\frac{L^A_u(t'|x_{r,u})}{x_{r,u}(t')-t'}$. Solving this differential equation gives,

$$
\begin{aligned}
&Ln(L^A_u(t'|x_{r,u})) \\
=&\int_{\tilde{t}}^{t'} \frac{1}{x_{r,u}(t'') - t''} dt'' + Ln(L^A_u(\tilde{t}|x_{r,u})) \\
=&Ln\left(\frac{x_{r,u}(t) - t}{x_{r,u}(t) - \underline{\tau}(t)}\right) + \sum_{(\underline{t},\bar{t}) \in \tilde{J}^p : \bar{t} < t} Ln\left(\frac{x_{r,u}(\underline{t}) - \bar{t}}{x_{r,u}(\underline{t}) - \underline{t}}\right) \\
&+ \sum_{(\underline{t},\bar{t}) \in \tilde{J}^s : \underline{t} < t} \left(2/\rho\left((x_{r,u}(\underline{t}) - \underline{t}) - (x_{r,u}(\max\{\bar{t}, t\}) - \max\{\bar{t}, t\})\right)\right) + Ln(L^A_u(\tilde{t}|x_{r,u})).
\end{aligned}
$$

The second equality uses the fact that $x_{r,u}$ is constant on pooling intervals, and satisfies (2b) on separating intervals. Using the assumption that the desired equality holds for $\tilde{t}$ and by taking exponents delivers the result. *Q.E.D.*

Lemma 11. *Let $y \in C(t, u)$ and $\bar{t} \in \overline{\tau}(T)$ with $t < \bar{t} < t + \bar{\delta}$.*

$$y(\bar{t}) \geq \sqrt{u + \rho(R(t, \bar{t}) - t)} + t.$$

Proof. Suppose the inequality does not hold. Then, since y is increasing,

$$
\begin{aligned}
&\int_t^{\bar{t}} 2\left(y(t') - \sqrt{u + \rho(R(t, \bar{t}) - t)} - t\right) dt' < 0 \\
\iff& L^A(\bar{t}|y) > \left(\sqrt{u + \rho(R(t, \bar{t}) - t)} - (\bar{t} - t)\right)^2 + \rho(\bar{t} - R(t, \bar{t})).
\end{aligned}
$$

Note that since $\bar{t} - t < \bar{\delta}$, $y(t') > t'$ $\forall t' < \bar{t}$. Thus $y(\bar{t}) < \sqrt{u + \rho(R(t, \bar{t}) - t)} + t$ means that y gives $\bar{t}$ lower material loss than the pooling allocation. However this contradicts the last line of the display, because $r_y^*(\bar{t}) \geq R(t, \bar{t})$ since the pooling reputation is the minimum such reputation. *Q.E.D.*

Lemma 12. *Take any continuing allocation x at (t, u) such that $\bar{\delta} + t > \bar{t} \in \overline{\tau}(T)$.*

$$L^A(\bar{t}|x) > \left(\sqrt{u + \rho(\bar{t} - t)} - \bar{t}\right)^2.$$

Proof. I start by proving the claim below. The claim says that if the reputations for the agent are higher for any interval, then the agent experiences lower loss. In order to write the claim I will need to notate certain dependences on the distribution. I use subscripts in the agent loss and reputation function to denote dependence on a distribution f – L_f^A and R_f respectively.

Claim 1. *Let $\overline{f}$ strictly monotone likelihood ratio dominate f.*[37] *Let x be a continuing allocation at (t, u) under distribution f with $\bar{\delta} + t \geq \bar{t} \in \overline{\tau}_x(T)$. Let $\overline{x} \in C(t, u)$ under $\overline{f}$ such that $\overline{x}$ and $\underline{x}$ induce the same interval partition, i.e. $\overline{\tau}_x(t') = \overline{\tau}_{\overline{x}}(t')$ $\forall t'$.*[38]

$$L_{\overline{f}}^A(t'|\overline{x}) \leq L_f^A(t'|x) \ \forall t' \in [t, \bar{t}].$$

Proof of Claim: Note that $\forall t_1 < t_2$ $R_{\overline{f}}(t_1, t_2) > R_f(t_1, t_2)$ by the definition of MLR dominance. Note that at $t' = t$, $L_{\overline{f}}(t'|\overline{x}) = L_f(t'|x) = u$ by the fact that these are both continuing allocations.

Suppose the claim does not hold, i.e. $\exists t' \in [t, \bar{t}]$: $L_{\overline{f}}^A(t'|\overline{x}) > L_f^A(t'|x)$. Because the reputations are higher under $\overline{x}$, it must be that the material loss is lower under x, i.e. $x(t'^-) < \overline{x}(t'^-)$. By Lemma 8, these losses are both left and right differentiable in t', and

$$\frac{d}{dt^-}\left(L_f^A(t'|x) - L_{\overline{f}}^A(t'|\overline{x})\right) > 0$$
$$\iff x(t'^-) < \overline{x}(t'^-).$$

But this implies that $L^A(t''|x) < L^A(t''|\overline{x})$ $\forall t'' \in [t, t']$ which contradicts $L_{\overline{f}}^A(t|\overline{x}) = L_f^A(t|x) = u$. This proves the claim

Let $\overline{f}$ be a limiting MLRP dominating distribution such that $R_{\overline{f}}(t_1, t_2) = t_2$ $\forall t_1 \leq t_2$. Let $\overline{x}$ be the corresponding continuing allocation at (t, u) under $\overline{f}$ defined by having the same

[37] That is, $\forall t' > t$ $\frac{\overline{f}(t')}{f(t')} > \frac{\overline{f}(t)}{f(t)}$.

[38] Such a continuing allocation exists by the same arguments as in Lemma 8.

endpoints, i.e. $\overline{\tau}_x(t') = \overline{\tau}_{\overline{x}}(t')\ \forall t'$. Because of the previous claim, $L^A_{\overline{f}}(\overline{t}|\overline{x}) \leq L^A_f(\overline{t}|x)$. Now define an alternative continuing allocation z at (t,u) under distribution $\overline{f}$ that pools types below $\overline{t}$, i.e. it satisfies $z(t') \equiv \sqrt{u+\rho(\overline{t}-t)}+t\ \forall t' < \overline{t}$. Notice that under $\overline{f}$, $r^*_z(\overline{t}) = \overline{t} = r^*_x(\overline{t})$. By Lemma 11, $\overline{t} \leq z(\overline{t}) \leq x(\overline{t})$. Thus $z(t)$ delivers lower distortion and the same reputation so $(\sqrt{\rho(\overline{t}-t)+u}-\overline{t})^2 \leq L^A_{\overline{f}}(\overline{t}|\overline{x})$.

Q.E.D.

Lemma 13. *Consider an optimal allocation x^* with $\underline{t} \equiv \underline{\tau}(t) < \overline{\tau}(t) \equiv \overline{t}$ for some t. Let $L^A(\underline{t}|x^*) \equiv u_0$, $L^A(\overline{t}|x^*) \equiv u_1$, $x^*(\underline{t}) \equiv \tilde{x}$, $R(\underline{t},\overline{t}) \equiv \tilde{R}$, and $F([\underline{t},\overline{t}]) \equiv \tilde{F}$. It holds that,*

$$\frac{2\tilde{F}(\tilde{x}-\tilde{R})}{\tilde{x}-\underline{t}+\sqrt{u_0}} - (\tilde{R}-\underline{t})f(\underline{t}) + \frac{F([\overline{t},M])}{\tilde{F}}V_{u^-}(t,u_1)\left(\frac{2\tilde{F}(\tilde{x}-\overline{t})}{\tilde{x}-\underline{t}+\sqrt{u_0}} - (\overline{t}-\underline{t})f(\underline{t})\right) \geq 0.$$

Proof. Consider the continuing allocation $x^*_{|[\underline{t},M]}$ at $(\underline{t},u_0)$. I will show that the inequality above results from this continuing allocation being optimal at (t,u_0). Consider an alternative continuing allocation at $(\underline{t},u_0)$ parameterized by m that (i) separates between $\underline{t}$ and m, (ii) pools between m and $\overline{t}$, and (iii) chooses an optimal continuing allocation above $\overline{t}$. Let $a^m \equiv \sqrt{D_{u_0}(m-\underline{t})+\rho(R(m,\overline{t})-m)}+m$ and $u^m \equiv (a^m-\overline{t})^2+\rho(\overline{t}-R(m,\overline{t}))$. Finally, let Let $x^*_{t,u}$ be an optimal continuing allocation at (t,u). Specifically the alternative continuing allocation is defined as

$$y^m \equiv \begin{cases} d_{u_0}(t'-\underline{t})+\underline{t} & \underline{t} \leq t' < m \\ a^m & m \leq t' < \overline{t} \\ x^*_{\overline{t},u^m}(t') & t' \geq \overline{t} \end{cases}.$$

Because x^* is optimal, it must be that $\frac{d\,L^P(y^m)}{dm}\big|_{m=\underline{t}} \geq 0$. This gives the inequality in the display. *Q.E.D.*

B. Proofs from Section 2

B.1. Proof of Lemma 1

Proof. " $\implies$ " $x(t)$ is increasing because $\forall x: T \to A$, $L^A(a,t|x)$ is strictly submodular in (a,t), so any minimizing selection from $\min_{a\in x(T)} L^A(a,t|x)$ is increasing.

Since $x(t)$ is increasing it has an at most countable set of discontinuities by Froda's theorem, i.e. J_x is countable.

Now suppose J_x is dense on some interval $[\underline{t},\overline{t}]$. First, consider that $x(t_1) = x(t_2)$ for $\underline{t} \leq t_1 < t_2 \leq \overline{t}$. Then since x is increasing $x(t)$ is constant on $[t_1,t_2]$, i.e. $J_x \cap (t_1,t_2) = \emptyset$

contradicting the hypothesis that J_x is dense on this interval. Thus, x is injective on $[\underline{t}, \bar{t}]$, and $r_x^*(t) = t\ \forall t \in [\underline{t}, \bar{t}]$. Note that $\forall x \in IC(T)$, $L^A(a, t|x)$ is continuous in t. Therefore, since the reputation is continuous on $[\underline{t}, \bar{t}]$ so is x, IC requires that the action be continuous on $(\underline{t}, \bar{t})$. This means $(\underline{t}, \bar{t}) \cap J_x = \emptyset$, which is a contradiction. This continuity of $L^A(t|x)$ also implies the third condition in the lemma.

This means J_x is a countable nowhere dense set. Take arbitrary $\underline{t}, \bar{t} \in J_x$ with $(\underline{t}, \bar{t}) \cap J_x = \emptyset$. First suppose that x is not strictly increasing on $(\underline{t}, \bar{t})$ Then since x is increasing there exists $\underline{t} \leq t_1 < t_2 \leq \bar{t}$ such that $x(t') \equiv \tilde{x}$ is constant $\forall t' \in (t_1, t_2)$, where $t_1 \equiv \inf\{t \in T : \ x(t) = \tilde{x}\}$ and $t_2 \equiv \sup\{t \in T : \ x(t) = \tilde{x}\}$. Suppose $t_1 > \underline{t}$. Then $\forall \varepsilon > 0$, $r_x(\tilde{x}) - r_x(x(t_1 - \varepsilon)) > R(t_1, t_2) - t_1$. Since $L^A(t|x)$ is continuous in t and the reputation discontinuously jumps at t_1, the action must also jump to preserve IC. But this contradicts the fact that $(\underline{t}, \bar{t}) \cap J_x = \emptyset$, so $t_1 = \underline{t}$. A symmetric argument shows that $t_2 = \bar{t}$, and so x is constant on $(\underline{t}, \bar{t})$. This means that x is either constant or strictly increasing on $(\underline{t}, \bar{t})$.

Suppose x is strictly increasing on $(\underline{t}, \bar{t})$. Then x is injective on this interval and $r_x(x(t)) = t\ \forall t \in (\underline{t}, \bar{t})$. And so

$$L^A(x(t'), t|x) = (x(t') - t)^2 + \rho(t - t').$$

For $\varepsilon > 0$, incentive compatibility implies,

$$\begin{aligned} & L^A(x(t+\varepsilon), t|x) - L^A(x(t), t|x) \geq 0 \\ \implies & x(t+\varepsilon)^2 - x(t)^2 - 2t(x(t+\varepsilon) - x(t)) \geq \rho\varepsilon \\ \implies & \frac{x(t+\varepsilon) - x(t)}{\varepsilon} \geq \frac{\rho}{x(t) + x(t+\varepsilon) - 2t}. \end{aligned}$$

An analogous argument for $\varepsilon < 0$ applies, so it holds that $\forall \varepsilon > 0$,

$$\frac{\rho}{x(t) + x(t-\varepsilon) - 2t} \geq \frac{x(t+\varepsilon) - x(t)}{\varepsilon} \geq \frac{\rho}{x(t) + x(t+\varepsilon) - 2t}.$$

Since $x(t+\varepsilon)$ is continuous in ε for $|\varepsilon|$ small. $x'(t) = \frac{\rho}{2(x(t)-t)}$.

" $\Longleftarrow$ " Since $L^A(a, t|x)$ is submodular, local incentive constraints are sufficient for global incentive constraints.[39] First consider $t \notin J_x$. Either x is constant around t or solves (2b) which was shown to preserve local incentives. For $t \in J_x$ condition 3 is equivalent to local incentive compatibility. *Q.E.D.*

[39] This fact is detailed in Carroll (2012).

B.2. Proof of Lemma 2

Proof. The solution to (2b) exists and is given by

$$d_u(t) \equiv t + \rho/2 \left(1 + W_0 \left(-e^{-\frac{\rho-2(\sqrt{u}-t)}{\rho}} \frac{\rho - 2\sqrt{u}}{\rho}\right)\right),$$

Where $W_0(z)$ is the Lambert W-function, i.e. the principal solution to $z = W_0(z)e^{W_0(z)}$. The properties of the lemma come directly from examining (2b).

Q.E.D.

B.3. Proof of Lemma 3

Proof. Note that by Lemma 7, it is without loss to take $A \equiv [0, B]$. Take an IC allocation $x : T \to \mathbb{R}$. By the envelope theorem, $L_t^A(t|x) = \rho - 2(x(t) - t)$. Notice that this derivative is uniformly bounded because $(x(t), t)$ are from a compact set. Take the set of loss functions induced by IC allocations to be $\mathcal{L} \equiv \{L^A(t|x) : x \in IC(T)\}$. $\mathcal{L}$ is a set of uniformly equicontinuous functions and is thereby compact by the Arzela – Ascoli theorem. Since $L^P(x) = \int_0^M L^A(t'|x)dt'$, the principal's loss is continuous in the agent's loss and a minimum exists.

Q.E.D.

C. Proofs from Section 3

C.1. Proof of Lemma 4

Proof. By the same argument as in Lemma 7, all continuing allocations y at (t, u) have $y(t') < \sqrt{u + \rho(M - t) + (M - t)^2} + M \;\; \forall t' \in T$. Thus $C(t, u)$ is a compact set and an optimal continuing allocation exists by the same argument as in Lemma 3. *Q.E.D.*

C.2. Proof of Proposition 1

Proof. Take $t \geq 0$, and $u \geq 0$, and let x^* be an optimal continuing allocation at (t, u). Define $\tilde{L}(t') \equiv (x^*(t') - t')^2 - R(t', \overline{\tau}(t'))$. This is the agent's loss using the x^* allocation but the higher reputation for a left censored interval. Let $u_0 = u - \varepsilon$ and $t_0 \equiv t$. Construct a set of sequences $\{u_i\}, \{t_i\}$ as follows. Let t_{i+1} be the first type $t' \geq \overline{\tau}(t_i)$ to solve the equation

$$\tilde{L}(t') = D_{u_i}(t' - \overline{\tau}(t_i)).$$

If no such solution exists, let $t_{i+1} \equiv M$. Now set $u_{i+1} \equiv (x^*(\overline{\tau}(t_{i+1})^-) - \overline{\tau}(t_{i+1}))^2 + \rho(\overline{\tau}(t_{i+1}) - R(t_{i+1}, \overline{\tau}(t_{i+1})))$. Notice that if $\tilde{L}(\overline{\tau}(t_i)) = L^A(\overline{\tau}(t_i)|x^*) > u_i$, then $\tilde{L}(\overline{\tau}_{i+1}) = L^A(\overline{\tau}(t_{i+1})|x^*) > u_{i+1}$. Also by assumption, $\tilde{L}(\overline{\tau}(t_0)) = \tilde{L}(t) > u_0$. This means that the LHS starts above the RHS at any iteration. Also $\tilde{L}(t')$ is continuous for $t' \notin J_{x^*}$. Moreover $\forall \tilde{t} \in J_{x^*}$,

$$\lim_{t' \to \tilde{t}^-} \tilde{L}(t') \leq \lim_{t' \to \tilde{t}^-} L^A(t'|x^*) = \lim_{t' \to \tilde{t}^+} L^A(t'|x^*) = \lim_{t' \to \tilde{t}^+} \tilde{L}(t'),$$

where the first and third relations follow from the definition $\tilde{L}$, and the second relation is from Lemma 1. Thus $\tilde{L}$ is continuous except at upwards jumps, and so $\tilde{L}(t') > D_{u_i}(t')\ \forall t' < t_{i+1}$.

Now define a continuing allocation y at $(t, u - \varepsilon)$ as

$$y(t') = \begin{cases} d_{u_i}(t' - \overline{\tau}(t_i)) + \overline{\tau}(t_i) & t' \in [\overline{\tau}(t_i), t_{i+1}) \\ x^*(t') & \text{otherwise} \end{cases}.$$

This allocation satisfies incentive compatibility by construction. Note that $\forall t' \in [t, M]$ $L^A(t'|x) \geq L^A(t'|y)$. Now let $K \equiv \inf_{0 \leq t_1 \leq t_2 \leq M} (R(t_1, t_2) - t_1) \frac{f(t_1)}{F([t_1, t_2])}$ be the minimum derivative of the reputation for an interval $R(t_1, t_2)$ with respect to its lower endpoint. Note that because $f(t) \in [\underline{k}, \overline{k}]$ by assumption, $K > 0$.

$$\begin{aligned}
&\frac{V(t,u) - V(t, u-\varepsilon)}{\varepsilon} \\
\geq &\frac{\int_t^M ((x^*(t') - t')^2 - (y(t) - t)^2)\, f(t) dt'}{\varepsilon} \\
= &\frac{\int_t^M \left(L^A(t'|x^*) - L^A(t'|y)\right) f(t) dt'}{\varepsilon} \\
\geq &\frac{\int_t^{t_1} \left(L^A(t'|x^*) - L^A(t'|y)\right) f(t) dt'}{\varepsilon} \\
= &\frac{\varepsilon - \int_t^M 2(x^*(t') - y(t')) F([t', M]) dt'}{\varepsilon} \\
\geq &1 - \frac{\int_t^{t_1} 2(x^*(t') - y(t')) F([t', M]) dt'}{\int_t^{t_1} 2(x^*(t') - y(t')) + \rho(R(t', \overline{\tau}(t')) - t') \frac{f(t')}{F((t', \overline{\tau}(t')))} dt'} \\
\geq &1 - \sup_{t \leq t' \leq t_1} \frac{2(x^*(t') - y(t')) F([t', M])}{2(x^*(t') - y(t')) + \rho(R(t', \overline{\tau}(t')) - t') \frac{f(t')}{F((t', \overline{\tau}(t')))}} \\
\geq &1 - \frac{2B}{2B + \rho K}.
\end{aligned}$$

The first line is due to the fact that x^* is optimal under (t, u) while y is feasible under $(t, u - \varepsilon)$. The second line is due to the fact that reputations integrate out to $\rho\mathbb{E}[t']$ for any allocation. The third line uses the fact that $\forall t' \in [t, M]$ $L^A(t|x) \geq L^A(t'|y)$. The fourth line uses integration by parts and Lemma 8. The fifth line uses the fact that $\tilde{L}(t_1) \geq D_{u-\varepsilon}(t_1)$ and $\tilde{L}(t) = D_{u-\varepsilon}(0) + \varepsilon$, so the change in losses must be less than ε over this interval, i.e. $\int_t^{t_1} 2(x^*(t') - y(t')) + \rho(R(t', \overline{\tau}(t')) - t')\frac{f(t')}{F((t', \overline{\tau}(t')))}dt' \leq \varepsilon$. The last two lines use the Cauchy mean value theorem and the bounds on the action and the derivative of the reputation function. *Q.E.D.*

C.3. Proof of Theorem 1

Proof. Suppose the theorem does not hold, i.e. there is a sequence of allocations x_n, each one optimal, such that $\lim_{n\to\infty} x_n(0) = 0$. Let $\underline{\tau}_n$ and $\overline{\tau}_n$ be the associated endpoint functions. Note that $\lim_{n\to\infty} \overline{\tau}_n(0) = 0$ as well. This is because $x_n(0) \geq R(0, \overline{\tau}_n(0))$. Otherwise, an alternative allocation that increases $x_n(0)$ to $R(0, \overline{\tau}_n(0))$ improves the principal's loss on the first interval $[0, \overline{\tau}_n(0)]$, and the initial loss of the agent at type $\overline{\tau}_n(0)$. By the alignment principle, this latter change also improves the principal's loss. I begin by breaking the problem into two exhaustive cases described below.

Case 1: There exists a pair of subsequences $t_n, s_n \in \overline{\tau}_n(T)$ and $b > 0$ such that $s_n > b$, $(t_n, s_n) \cap \overline{\tau}_n(T) = \emptyset$ $\forall n$, and as $n \to \infty$, $t_n \to 0$.

Case 2: There exists a pair of subsequences $t_n, s_n \in \overline{\tau}_n(T)$ such that as $n \to \infty$, $t_n \to 0$, $s_n \to 0$, and $\frac{t_n}{s_n - t_n} \to 0$.

Claim 2. *Either case 1 or case 2 holds.*

Proof of Claim: Take a subsequence $t_n < 1/n^3$ $\forall n$ which exists by the fact that $\lim_{n\to\infty} \overline{\tau}_n(0) = 0$. Now suppose that there exists a subsequence s_{n_k} such that $s_{n_k} \in \overline{\tau}_{n_k}(T) \cap (1/n^2, 1/n)$. Then this pair of subsequences satisfies case 2. Suppose that there exists no such subsequence, i.e. $\forall n > N$, $\overline{\tau}_n(T) \cap (1/n^2, 1/n) = \emptyset$. Then take $r_n \equiv \max \overline{\tau}_n(T) \cap [0, 1/n^2]$ and $s_n \equiv \min\ [1/n, M] \cap \overline{\tau}_n(T)$. The subsequences r_n and s_n satisfy case 1 in the case that $s_n \not\to 0$, and case 2 otherwise. This proves the claim.

Suppose case 1 holds. This means that for the relevant subsequences $t_n, s_n \in \overline{\tau}_n(T)$, $\exists b > 0 :\ s_n > b$ $\forall n$. Let $u_n \equiv L^A(s_n|x_n)$ and $a_n \equiv x_n(t_n)$. Applying Lemma 13 to (t_n, s_n)

gives

$$\frac{2}{x_n(t_n) - t_n + \sqrt{L^A(t_n|x_n)}} \left((a_n - R(t_n, s_n))F([t_n, s_n]) + (a_n - s_n)V_{u^-}(s_n, L^A(s_n|x_n))F([s_n, M])\right)$$
$$\geq f(t_n) \left((R(t_n, s_n) - t_n) + (s_n - t_n)\frac{F([s_n, M])}{F([t_n, s_n])}V_{u^-}(s_n, L^A(s_n|x_n))\right). \tag{10}$$

Now let $\tilde{u}_n \equiv (a - s_n)^2 + \rho(s_n - R(0, s_n))$ for $a \in A$, and consider the alternative allocation

$$x_n^a(t) \equiv \begin{cases} a & t' < s_n \\ x^*_{s_n,\tilde{u}_n}(t') & t \geq s_n \end{cases}.$$

The change in $L^P(x_n^a)$ from changing a is given by,

$$(a - R(0, s_n))F([0, s_n]) + (a - s_n)F([s_n, M])V_{u^-}(s_n, \tilde{u}_n).$$

Given (10), and the fact the $t_n \to 0$, setting $a = x_n(t_n)$ makes the above derivative positive and bounded away from 0 across n. This uses that (i) since $s_n > b$, $x_n(t_n) = \sqrt{L^A(t_n|x_n) + \rho(R(t_n, s_n) - t_n)} + t_n$ is bounded away from 0, (ii) $R(t_n, s_n) - t_n$ is bounded away from 0, and (iii) $V_{u^-}(s_n, L^A(s_n|x_n)) > 0$. Let a^* be the optimizing action in x_n^a. This means that $L^P(x_n^{x_n(t_n)}) - L^P(x_n^{a^*})$ is bounded away from 0. However, since $t_n \to 0$, $L^P(x_n^{x_n(t_n)}) - L^P(x_n) \to 0$ as $n \to \infty$. This contradicts the fact that x_n is optimal for large n.

Now consider that Case 2 holds. There exists a pair of subsequences $t_n, s_n \in \tau_n(T)$ such that as $n \to \infty$, $t_n \to 0$, $s_n \to 0$, and $\frac{t_n}{s_n - t_n} \to 0$. By Lemma 12,

$$L^A(s_n|x_n) \geq \left(\sqrt{L^A(t_n|x_n) + \rho(s_n - t_n)} - (s_n - t_n)\right)^2 \equiv u_n.$$

By the alignment principle, this means that $L^P(x_n) \geq \int_0^{s_n}(x_n(t') - t')^2 f(t')dt' + F([s_n, M])V(s_n, u_n)$. Now construct an alternate sequence of allocations z_n defined by

$$z_n(t) \equiv \begin{cases} s_n & t < s_n \\ x^*_{s_n,\rho(s_n - R(0,s_n))}(t) & t \geq s_n \end{cases}.$$

The loss from z_n is $L^P(z_n) = \int_0^{s_n}(s_n - t')^2 f(t')dt' + F((s_n, M))V(s_n, \rho(s_n - R(0, s_n)))$. Notice that because $\frac{t_n}{s_n - t_n} \to 0$, $u_n - \rho(s_n - R(0, s_n)) \sim \rho s_n/2$ for large n, i.e. it is approximately linear in n. Because of the alignment principle, this means that the difference in loss between x_n and z_n for types in $[s_n, M]$ is positive and linear in s_n. Moreover, because $s_n \to 0$ the difference in loss between x_n and z_n on $[0, s_n]$ is second order. This means that for large n,

z_n does better than x_n contradicting the latter allocation's optimality. This completes the proof for case 2 and thereby the full argument. *Q.E.D.*

D. Proofs from Section 4

D.1. Proof of Lemma 5

Proof. Note that $\overline{u}(t,u) = u + \rho t - 2t\sqrt{u + \rho R(t)} + t^2$. Since $\overline{u}(t,u)$ is decreasing in $R(t)$, it suffices to prove the lemma in the uniform limit where $R(t) = t/2$. Observe that $\overline{u}(t,u)$ is continuously differentiable in t $\forall u, t$. In addition, $\forall u$ $\overline{u}(0,u) - u = 0$. This means that in order to prove the lemma I only need to establish that $\overline{u}(t,u) - u$ is strictly single crossing from below in t for $u \le \rho^2/4$, and that $\overline{u}(t,u) - \rho^2/4$ is strictly single crossing from below for $u \ge \rho^2/4$.

First suppose $u < \rho^2/4$. Assume $t > 0$ and $\overline{u}(t,u) = u$, i.e.

$$\rho t - 2t\sqrt{u + \rho t/2} + t^2 = 0. \tag{11}$$

I next evaluate the derivative of $\overline{u}(t,u)$ with respect to t.

$$\begin{aligned}
\frac{d(\overline{u}(t,u))}{dt} &= \rho - 2\sqrt{u + \rho t/2} + 2t - \frac{\rho t/2}{\sqrt{u + \rho t/2}} \\
&= t - t\left(\frac{\rho/2}{\sqrt{u + \rho t/2}}\right) \\
&= \frac{1}{2\sqrt{u + \rho t/2}}\left(2t\sqrt{u + \rho t/2} - \rho t\right) \\
&= \frac{t^2}{2\sqrt{u + \rho t/2}} > 0.
\end{aligned}$$

The first and last equality use (11). Note that for $t = 0$, $\overline{u}_t(0,u) = \rho - 2\sqrt{u}$ which is positive if $u < \rho^2/4$.

Now suppose $u \ge \rho^2/4$.

$$\overline{u}(t,u) \ge \rho^2/4 \tag{12}$$

$$\iff \rho t - 2t\sqrt{u + \rho t/2} + t^2 + u > \rho^2/4. \tag{13}$$

note that the LHS of (13) is quasiconvex in u with a minimum at $u = \max\{\rho^2/4, t^2 - \rho t/2\}$.

If $\max\{\rho^2/4, t^2 - \rho t/2\} = \rho^2/4 = u$, then (13) becomes,

$$\rho - 2\sqrt{\rho^2/4 + \rho t/2} + t > 0$$

which holds for all $t > 0$ as the LHS is strictly convex in t with a 0 derivative at $t = 0$. If instead $\max\{\rho^2/4, t^2 - \rho t/2\} = t^2 - \rho t/2 = u$, then the LHS of (13) evaluates to $\rho t/2 > 0$. This completes the proof. *Q.E.D.*

D.2. Proof of Proposition 2

Proof. I begin by proving the following approximation claim on the principal's and agent's loss for small intervals.

Claim 3. *Take any continuing allocation x at $u \geq 0$ with a threshold at $\delta \in \overline{\tau}(T)$ with $\delta < \bar{\delta}$. Let z be an alternative pooling continuing allocation with first threshold δ, i.e. it satisfies $z(t') = \sqrt{u + \rho R(\delta)}\ \forall t' \in [0, \delta]$. It holds that*

$$\begin{aligned}
&\frac{\int_0^\delta \left((z(t') - t')^2 - (x(t') - t')^2\right) \lambda e^{-\lambda t'} dt'}{1 - e^{-\lambda\delta}}. \\
\leq &\max\{0, L^A(\delta|z) - L^A(\delta|x)\} && (14) \\
\leq &2\delta^{3/2}\sqrt{\rho} && (15)
\end{aligned}$$

Proof of Claim: Note that,

$$\begin{aligned}
&\frac{\int_0^\delta \left((z(t') - t')^2 - (x(t') - t')^2\right) \lambda e^{-\lambda t'} dt'}{1 - e^{-\lambda\delta}} \\
= &\frac{\int_0^\delta \left(L^A(t'|z) - L^A(t'|x)\right) \lambda e^{-\lambda t'} dt'}{1 - e^{-\lambda\delta}} \\
\leq &\max_{t' \leq \delta} L^A(t'|z) - L^A(t'|x),
\end{aligned}$$

where the last inequality is due to the Cauchy mean value theorem. By Lemma 8 the derivative of $L^A(t'|z) - L^A(t'|x)$ with respect to t' is given by $2(x(t') - z(t'))$. Since x is increasing and z is constant, the integrand is convex in t' and thereby maximized at one of its endpoints. Because both are continuing allocations at u, $L^A(0|z) - L^A(0|x) = 0$, which delivers the first inequality. By Lemma 12

$$
\begin{aligned}
&L^A(\delta|z) - L^A(\delta|x) \\
\leq &\left(\sqrt{u+\rho R(\delta)} - \delta\right)^2 + \rho(\delta - R(\delta)) - \left(\sqrt{u+\rho\delta} - \delta\right)^2 \\
= &2\delta\left(\sqrt{u+\rho\delta} - \sqrt{u+\rho R(\delta)}\right) \\
\leq &2\delta^{3/2}\sqrt{\rho}.
\end{aligned}
$$

The last inequality follows from concavity of the square root function. This proves the claim.

Suppose that for some $u \geq 0$, $L^P(y_u) > V(u) + 2\varepsilon$ for some small $\varepsilon > 0$. Let y^* be a continuing allocation at u that achieves $L^P(y^*) < V(u) + \varepsilon$. Suppose first that there is a highest threshold $t = \max\ \underline{\mathcal{T}}_{y^*}(T)$ under y^*, i.e. y^* is pooling after t. Then note that

$$
\begin{aligned}
&\frac{\int_t^\infty \lambda(y^*(t') - t')^2 e^{-\lambda t'} dt'}{e^{-\lambda t}} \\
= &\lim_{\bar{t}\to\infty} \int_0^{\bar{t}} \lambda(a(\bar{t}, u) - t')^2 e^{-\lambda t'} dt' + e^{-\lambda \bar{t}} L^P(y_{\overline{u}(\bar{t},u)}) \\
\geq &L^P(y_{L^A(t|y^*)})
\end{aligned}
$$

where the equality uses $\frac{d\,L^P(y_u)}{du} \leq 1$ and the inequality is by Lemma 5 and the assumption that y_u satisfies the Bellman equation. In this case set $t_0 = t$.

Now suppose that $\underline{\mathcal{T}}_{y^*}(T)$ is unbounded. There must exist a threshold $t \in \underline{\mathcal{T}}(T)$ for which $L^P(y_{L^A(t|y^*)})e^{-\lambda t} < \varepsilon/2$. If not, since $\frac{d\,L^P(y_u)}{du} \leq 1$, this would imply that $L^P(y^*) = \infty$. Take such a t and set $t_0 = t$.

Take arbitrary $\delta \in (0, \bar{\delta})$. I will iteratively revise y^* at a sequence of thresholds decreasing from t_0 to 0 in finite steps and end with y_u. Define t_{i+1} recursively as follows.

Case 1: there exists a first threshold $s \in \underline{\mathcal{T}}_{y^*}(T)$ such that $s \leq t_i - \delta$, i.e. $\underline{\mathcal{T}}_{y^*}(t') = s\ \forall t' \in [s, t_i)$, in which case define $t_{i+1} \equiv s$.

Case 2: case 1 does not hold, in which case define $t_{i+1} = \min([t_i - \delta, t_i] \cap \underline{\mathcal{T}}_{y^*}(T))$.

Note that one of the two cases must hold, and that in either case $t_i - t_{i+2} > \delta\ \forall i$. Now define,

$$
y_i(t') \equiv \begin{cases} y^*(t') & t' < t_i \\ y_{L^A(t_i|y^*)}(t' - t_i) & t' \geq t_i \end{cases}.
$$

By construction the loss from changing from y^* to y_0 is less than $\varepsilon/2$. I now analyze the change in loss from y_i to y_{i+1}. Consider the continuing allocation z at $L^A(t_{i+1}|y^*)$ defined by

$$z(t) \equiv \begin{cases} a(t_i - t_{i+1}, L^A(t_{i+1}|y^*)) & t' < t_i - t_{i+1} \\ y_{\overline{u}(t_i - t_{i+1}, L^A(t_{i+1}|y^*))}(t' - t_i) & t' \geq t_i - t_{i+1} \end{cases}.$$

Allocation y_i uses y^* below t_i and the associated y_u above t_i. Allocation z pools t_{i+1} to t_i and then uses the appropriate y_u above t_i. The losses from z and y_i above t_{i+1} are given respectively by

$$\int_{t_{i+1}}^{t_i} \lambda \left(a(t_i - t_{i+1}, L^A(t_{i+1}|y^*)) - (t' - t_{i+1})\right)^2 e^{-\lambda t'} dt' + L^P(y_{\overline{u}(t_i - t_{i+1}, L^A(t_{i+1}|y^*))}) e^{-\lambda t_i}, \quad (16)$$

$$\int_{t_{i+1}}^{t_i} \lambda (y^*(t') - t')^2 e^{-\lambda t'} dt' + L^P(y_{L^A(t_i|y^*)}) e^{-\lambda t_i}. \quad (17)$$

If t_{i+1} is chosen according to case 1, then $z(t - t_{i+1}) = y_i(t)\ \forall t \geq t_{i+1}$ and the two losses are equivalent. If case 2 holds instead, then $t_i - t_{i+1} < \delta$. Now note that the difference between (16) and (17) is

$$\begin{aligned}
&\frac{\int_0^{t_i - t_{i+1}} \left(\left(a(t_i - t_{i+1}, L^A(t_{i+1}|y^*)) - t'\right)^2 - (y^*(t' + t_{i+1}) - (t' + t_{i+1}))^2 \right) \lambda e^{-\lambda t'} dt'}{1 - e^{-\lambda(t_i - t_{i+1})}} (e^{-\lambda t_{i+1}} - e^{-\lambda t_i}) \\
&+ \left(L^P(y_{\overline{u}(t_i - t_{i+1}, L^A(t_{i+1}|y^*))}) - L^P(y_{L^A(t_i|y^*)}) \right) e^{-\lambda t_i} \\
\leq& 2\sqrt{\rho}(t_i - t_{i+1})^{3/2}(e^{-\lambda t_{i+1}} - e^{-\lambda t_i}) + 2\sqrt{\rho}(t_i - t_{i+1})^{3/2} e^{-\lambda t_i} \\
\leq& 2\sqrt{\rho}(t_i - t_{i+1})^{3/2} \\
\leq& 2\delta^{3/2}\sqrt{\rho}. \qquad (18)
\end{aligned}$$

The first inequality applies Claim 3 twice and the fact that $\frac{d\, L^P(y_u)}{du} \leq 1$. The loss from y_{i+1} above t_{i+1} is $L^P(y_{L^A(t_{i+1}|y^*)}) e^{-\lambda t_{i+1}}$ which is less than that of z by Lemma 5 and the assumption that y_u satisfies the Bellman equation. Thus the expression in (18) is actually an upper bound on the difference in loss between y_{i+1} and y_i.

Note that because $t_i - t_{i+2} > \delta$ there exists $n < 2t_0/\delta$ such that $y_n = y_u$. Thus the total change in loss at the end of the process is bounded above by the sum over (18) given by

$$\begin{aligned} &n\delta^{3/2}\sqrt{\rho} \\ \leq& 4t_0\delta^{1/2}\sqrt{\rho} \end{aligned}$$

which goes to 0 with δ. Therefore, by choosing δ small enough, one can guarantee that this sum is less than $\varepsilon/2$, which is a contradiction to the hypothesis that $L^P(y_u) > V(u) +$

2ε. *Q.E.D.*

D.3. Proof of Theorem 2

It will be convenient to notate the principal's separating loss as $L^P(d_u) \equiv V_s(u)$.

Proof. First I prove some claims about the first and second derivatives of $V_s(u)$.

Claim 4. *The separating allocation satisfies the following.*

1. $V_s'(u) = \begin{cases} \int_0^\infty \frac{\rho - 2\sqrt{D_u(t')}}{\rho - 2\sqrt{u}} \lambda e^{-\lambda t'} dt' & u \neq \rho^2/4 \\ \frac{\rho\lambda}{\rho\lambda + 2} & u = 1/4 \end{cases}.$
2. $V_s'(u) = \lambda \frac{V_s(u) - u}{\rho - 2\sqrt{u}} \quad \forall u \neq \rho^2/4.$
3. $V_s'(u) \leq 1 \quad \forall u \geq 0.$

Proof of Claim: The second point is obtained from the definition of $V_s(u)$, the first point, and integration by parts:

$$\begin{aligned} V_s(u) &= \int_0^\infty D_u(t') \lambda e^{-\lambda t'} dt' \\ &= u + \int_0^\infty \left(\rho - 2\sqrt{D_u(t')}\right) e^{-\lambda t'} dt', \end{aligned}$$

where the second equality uses $D_u'(t') = \rho - 2\sqrt{D_u(t')}$ which follows from (2b).

The third point in the claim follows from inspecting the integrand in the first point and using the fact in Lemma 2 that $\rho - 2\sqrt{u} \geq (\leq) 0 \implies \rho^2/4 \geq (\leq) D_u(t) \geq (\leq) u$.

To see the first point, note that because of (2b), $\forall t_2 \geq t_1 \geq 0$, $D_u(t_2) = D_{D_u(t_1)}(t_2 - t_1)$. Thus define $m_u(v)$ be the type t that solves $D_u(t) = v$. By Lemma 2, $m_u(v)$ is well defined for all u, v such that either $\rho^2/4 \geq v \geq u$ or $\rho^2/4 \leq v \leq u$. Using implicit differentiation gives $m_u'(v) = \frac{1}{\rho - 2\sqrt{D_u(m_u(v))}}$. Thus,

$$\frac{d\ D_u(t)}{du} = \frac{dD_u(m_u(u+\varepsilon)+t)}{d\varepsilon}\Big|_{\varepsilon=0} = \frac{\rho - 2\sqrt{D_u(t')}}{\rho - 2\sqrt{u}}.$$

The definition in point 1 for $u = \rho^2/4$ is obtained by using l'Hopital's rule on the expression in point 2. This proves the claim.

Claim 5. *The separating allocation satisfies the following.*

1. $V_s''(u) = \begin{cases} \int_0^\infty \frac{\rho-2\sqrt{D_u(t')}}{(\rho-2\sqrt{u})^2} \frac{2(\sqrt{D_u(t')}-\sqrt{u})}{\sqrt{u}\sqrt{D_u(t')}} \lambda e^{-\lambda t'} dt' & u \neq \rho^2/4 \\ \frac{4\lambda\rho}{(\lambda\rho+2)(\lambda\rho+4)} & u = \rho^2/4 \end{cases}$.

2. $V_s''(u) = \frac{V_s'(u)(\lambda+\frac{1}{\sqrt{u}})-\lambda}{\rho-2\sqrt{u}} \quad \forall u \neq \rho^2/4$

3. $V_s''(u) \geq 0 \quad \forall u \geq 0$

Proof of Claim: To see point 1, take the derivative of the expression in point 1 of Claim 4 in u. This uses the fact (established in the proof of that point) that $\frac{d(D_u(t))}{du} = \frac{\rho-2\sqrt{D_u(t)}}{\rho-2\sqrt{u}}$. To see point 2, take the derivative of both the left hand side and right hand side of point 2 of Claim 4. To get point 1 for the case of $u = \rho^2/4$, use l'Hopital's rule on the expression in point 2. Point 3 is seen by inspecting point 1 and noting that $\rho - 2\sqrt{u} \geq (\leq) 0 \implies \rho^2/4 \geq (\leq) D_u(t) \geq (\leq) u$.

Define $L_{u,\lambda} \equiv \{t : \; R(t) + \rho/4 < \sqrt{u + \rho R(t)}\}$.

Claim 6. *$t \in L_{u,\lambda} \implies d_1 + d_0 > \rho/2$. Moreover, for $\rho\lambda > 2$ and $\forall u \geq \rho^2/16$, $L_{u,\lambda} = [0, \infty)$.*

Proof of Claim: Suppose instead that $d_1 + d_0 \leq \rho/2$. By definition $(d_1 + d_0)(d_1 - d_0) = \rho(t - R(t))$. Using the assumed inequality gives

$$\begin{aligned} &\rho/2(\rho/2 - 2d_0) \geq \rho(t - R(t)) \\ \iff &\rho/2 - 2(a(t,u) - t) \geq 2(t - R(t)) \\ \iff &\rho/4 + R(t) \geq \sqrt{u + \rho R(t)} \\ \implies &t \notin L_{u,\lambda}. \end{aligned}$$

Next, if $\rho\lambda > 2$, then $R(t) < \rho/2$ which implies that $\forall u \geq \rho^2/16$, $L_{u,\lambda} = [0, \infty)$. This proves the claim.

I will confirm that

$$V_s(u) \leq \int_0^t \lambda(a(t,u) - t')^2 e^{-\lambda t'} dt' + e^{-\lambda s} V_s(\overline{u}(t,u)) \quad \forall t > 0, \; \forall u \geq \rho^2/16. \tag{19}$$

I will first show that there exists a large λ such that the condition holds for all $\lambda' > \lambda$. I will then show that this condition is tighter for higher λ, i.e. if it holds for some fixed λ then it holds for all lower λ. Define

$$\tilde{V}(t,u) \equiv \int_0^t \lambda(a(t,u) - t')^2 e^{-\lambda t'} dt' + e^{-\lambda t} V_s(\overline{u}(t,u)) \tag{20}$$

as the principal's loss from choosing first threshold t and separating thereafter given initial loss u.

Part 1: For any small $\varepsilon > 0$. There exists a large $\tilde{\lambda}$ such that $\forall \lambda > \tilde{\lambda}$, $\forall t > 0$, and $\forall u > \rho^2/16 + \varepsilon$, $\tilde{V}(t,u) - V_s(u) > 0$.

More specifically, for the same qualifiers, I will show that $\frac{d\,\tilde{V}(t,u)}{dt} \geq 0\ \forall t > 0$. Since $\tilde{V}(0,u) = V_s(u)$, this completes part 1. I begin by expanding and simplifying this derivative condition.

$$
\begin{aligned}
&\frac{d\left(\int_0^t \lambda(a(t,u)-t')^2 e^{-\lambda t'}dt' + e^{-\lambda t}V_s(\overline{u}(t,u))\right)}{dt} \\
=&\rho R'(t)\frac{(a(t,u)-R(t))(1-e^{-\lambda t})}{a(t,u)} + \left((a(t,u)-t)^2 - V_s(\overline{u}(t,u))\right)\lambda e^{-\lambda t} \\
&+ e^{-\lambda t}V_s'(\overline{u}(t,u))\overline{u}_t(t,u) \geq 0 \\
\iff& \rho\lambda e^{-\lambda t}(t-R(t))\frac{(a(t,u)-R(t))}{a(t,u)} + \left(\overline{u}(t,u) - \rho(t-R(t)) - V_s(\overline{u}(t,u))\right)\lambda e^{-\lambda t} \\
&+ e^{-\lambda t}V_s'(\overline{u}(t,u))\overline{u}_t(t,u) \geq 0 \\
\iff& -\rho(t-R(t))\frac{R(t)}{a(t,u)} + \left(\overline{u}(t,u) - V_s(\overline{u}(t,u))\right) \\
&+ V_s'(\overline{u}(t,u))/\lambda \overline{u}_t(t,u) \geq 0,
\end{aligned}
$$

where the second equality uses $R'(t) = (t-R(t))\frac{\lambda e^{-\lambda t}}{1-e^{-\lambda t}}$ and $\overline{u}(t,u) = (a(t,u)-t)^2 + \rho(t-R(t))$. Using Claim 4 gives,

$$
V_s'(\overline{u}(t,u))\frac{d}{dt}\overline{u}(t,u) = \lambda\frac{V_s(\overline{u}(t,u)) - \overline{u}(t,u)}{\rho - 2\sqrt{\overline{u}(t,u)}}\left(\rho - \rho\frac{tR'(t)}{a(t,u)} - 2(a(t,u)-t)\right).
$$

Plugging the above identity to the above inequality gives

$$
\begin{aligned}
&-\rho(t-R(t))\frac{R(t)}{a(t,u)} + \frac{V_s(\overline{u}(t,u)) - \overline{u}(t,u)}{\rho - 2\sqrt{\overline{u}(t,u)}}\left(-\rho\frac{tR'(t)}{a(t,u)} + 2(\sqrt{\overline{u}(t,u)} + t - a(t,u))\right) \geq 0 \\
\iff& -R(t) + \frac{V_s(\overline{u}(t,u)) - \overline{u}(t,u)}{\rho - 2\sqrt{\overline{u}(t,u)}}\left(\frac{2a(t,u)}{a(t,u) + \sqrt{\overline{u}(t,u)} - t} - \frac{t\lambda e^{-\lambda t}}{1-e^{-\lambda t}}\right) \geq 0.
\end{aligned}
$$

The last implication uses the identity $(\sqrt{\overline{u}(t,u)} + t - a(t,u))(\sqrt{\overline{u}(t,u)} + a(t,u) - t) = \rho(t - R(t))$. Reusing this identity and dividing both sides by $R(t)$ reduces the above inequality to,

$$\frac{V_s(d_1^2) - d_1^2}{\rho - 2d_1} \left(\frac{\frac{t}{R(t)}(2(d_0 + d_1) - \rho) + \rho}{(d_0 + d_1)^2} + \lambda \right) \geq 1, \tag{21}$$

where $d_1 \equiv \sqrt{\overline{u}(t,u)}$ and $d_0 \equiv a(t,u) - t$. The goal of this part is for any small $\varepsilon > 0$, to find a large λ such that $\forall u \geq \rho^2/16 + \varepsilon$, $\forall t \geq 0$, and $\lambda' > \lambda$, (21) holds. I complete the argument in a series of claims that establish this fact on a finite partition of $\{(u,t) \in [\rho^2/16 + \varepsilon, \infty) \times [0, \infty)\}$.

Claim 7. *There exists λ_0, such that $\forall \lambda > \lambda_0$, (21) holds for $t = 0$ $\forall u \geq \rho^2/16 + \varepsilon$.*

Proof of Claim: When $t = 0$ $\overline{u}(t,u) = u$ and $a(t,u) = \sqrt{u}$ so (21) reduces to

$$\frac{V_s(u) - u}{\rho - 2\sqrt{u}} \left(\frac{8\sqrt{u} - \rho}{4u} + \lambda \right) \geq 1. \tag{22}$$

Suppose first that $u \leq \rho^2/4$. I use integration by parts to rewrite

$$V_s(u) = u + 1/\lambda \int_0^\infty \left(\rho - 2\sqrt{D_u(t')} \right) \lambda e^{-\lambda t'} dt'.$$

Since $\rho - 2\sqrt{D_u(t)}$ is convex by (2b) if $u \leq \rho^2/4$, Jensen's inequality implies that $V_s(u) - u \geq 1/\lambda \left(\rho - 2\sqrt{D_u(1/\lambda)} \right)$. Thus (21) at $t = 0$ holds if

$$\frac{8\sqrt{u} - \rho}{4u} \left(\rho - 2\sqrt{D_u(1/\lambda)} \right) > 2 \left(\sqrt{D_u(1/\lambda)} - \sqrt{u} \right) \lambda$$

Taking the limit of both sides as $\lambda \to \infty$ using l'Hopital's rule on the RHS, this inequality reduces to,

$$\frac{8\sqrt{u} - \rho}{4u}(\rho - 2\sqrt{u}) > \frac{\rho - 2\sqrt{u}}{\sqrt{u}}$$
$$\iff u > \rho^2/16.$$

Moreover, this convergence is uniform across $u > \rho^2/16 + \varepsilon$, as the LHS of (22) is Lipschitz continuous in u on $[\rho/16 + \varepsilon, \rho^2/4]$ with a constant independent of λ for large λ. This is because both $V_s'(u)$and $\frac{8\sqrt{u}-\rho}{4\lambda u}$ are each Lipschitz continuous with constant independent of

$\lambda > \overline{\lambda}$, $\forall \overline{\lambda} > 0$, and the LHS of (22) can be written as

$$V_s'(u)\left(\frac{8\sqrt{u}-\rho}{4\lambda u}+1\right).$$

To see Lipschitz continuity of V_s', I generate a uniform upper and lower bound on $V_s''(u)$. The lower bound comes from point 3 of (5). To determine the upper bound note that $\forall u$, $\rho - 2\sqrt{u} \geq (\leq) 0 \implies \rho^2/4 \geq (\leq) D_u(t) \geq (\leq) u$. Maximizing the integrand of point 1 in Claim 5 over $D_u(t')$ given these constraints yields an upper bound on $V_s''(u)$ that is independent of λ and decreasing in u.[40] The specific bound is then given by this in conjunction with $u \geq \rho^2/16$. This proves the claim for $u \in [\rho^2/16 + \varepsilon, \rho^2/4]$.

By Claim 4, $V_s'(\rho^2/4) = \frac{\lambda\rho}{\lambda\rho+2}$. Plugging $\rho^2/4 = u$ shows that (22) holds strictly at $u = \rho^2/4$ $\forall \lambda > 0$. Now, I will show that the LHS of (22) single crosses 1 from below in u for $u \geq \rho^2/4$ $\forall \lambda$. To see this, note that (22) for $u \geq \rho^2/4$ is equivalent to both

$$V_s'(u) \geq \frac{4u\lambda}{8\sqrt{u}-\rho+4u\lambda}, \text{ and}$$
$$V_s(u)\frac{8\sqrt{u}-\rho+4u\lambda}{3u+4u^2\lambda} \leq 1$$

Assume a crossing at some $u \geq \rho^2/4$, i.e. that both of the above inequalities hold with equality. I will show the derivative of the LHS of the second inequality in u is negative. This condition is equivalent to,

$$\begin{aligned} V_s'(u)(8\sqrt{u}-\rho+4u\lambda) + (4/\sqrt{u}+4\lambda)V_s(u) &< 3 + 8u\lambda \\ \iff 4u\lambda + (4/\sqrt{u}+4\lambda)\frac{3u+4u^2\lambda}{8\sqrt{u}-\rho+4u\lambda} &< 3 + 8u\lambda \\ \iff 0 &< (3+4u\lambda)(4\sqrt{u}-\rho) \\ \iff u &> \rho^2/16, \end{aligned}$$

where the first equivalence uses the identities assumed by a crossing at u. This means that (22) holds for all $u > \rho^2/16$ proving the claim for large enough λ.

Claim 8. *Taken any $\Delta > 1$. There exists λ_1 such that $\forall \lambda > \lambda_1$, $\forall t > 0$, $\forall u > \rho^2/16 + \varepsilon$ with $d_1 + d_0 > \Delta\rho$ and $a(t,u) \geq t$,* (21) *holds.*

Proof of Claim: Take any u, t in the region and rewrite notation fixing $\overline{u}(t,u) \equiv \overline{u}$. With

[40] This maximization sets $D_u(t') = (\rho/2)\sqrt{u}\ \forall u$.

this new notation, $d_1 = \sqrt{\overline{u}}$ and $d_0 = \sqrt{\overline{u} - \rho(t - R(t))}$.[41] Thus rewrite (21) as

$$\frac{V_s(\overline{u}) - \overline{u}}{\rho - 2\sqrt{\overline{u}}} \left(\frac{\frac{t}{R(t)} \left(2\left(\sqrt{\overline{u} - \rho(t - R(t))} + \sqrt{\overline{u}}\right) - \rho\right) + \rho}{\left(\sqrt{\overline{u} - \rho(t - R(t))} + \sqrt{\overline{u}}\right)^2} + \lambda \right) \geq 1. \tag{23}$$

I will show that the LHS of (23) is increasing in t for any fixed $\overline{u}$, and is thereby tightest at $t = 0$ which is the case proved in Claim 7. The derivative of the LHS in t is positive if,

$$-\frac{\rho(1 - R'(t))}{R(t)\sqrt{\overline{u} - \rho(t - R(t))}} \frac{(t - R)\rho - t(\sqrt{\overline{u} - \rho(t - R(t))} + \sqrt{\overline{u}})}{(\sqrt{\overline{u} - \rho(t - R(t))} + \sqrt{\overline{u}})^3} \geq 0$$

The change in $t/R(t)$ is omitted because this term is increasing in t and $2(d_1 + d_0) - \rho$ is positive by Claim 6. The above inequality is positive because $(1 - R'(t)) < 0$ and $(t - R)\rho - t(\sqrt{u - \rho(t - R(t))} + \sqrt{u}) < 0$ based on the definition of this region. This proves the claim.

Claim 9. *Taken any* $\Delta > 1$*. There exists* λ_2 *such that* $\forall \lambda > \lambda_2$*,* $\forall t > 0$*,* $\forall u > \rho^2/16 + \varepsilon$ *with* $d_1 + d_0 > \Delta\rho$ *and* $a(t, u) < t$*, (21) holds.*

Proof of Claim: Similar to in Claim 8, let $\overline{u}(t, u) \equiv \overline{u}$ and note that $d_1 = \sqrt{\overline{u}}$ and $d_0 = -\sqrt{\overline{u} - \rho(t - R(t))}$.[42] Thus rewrite (21) as

$$\begin{aligned}
\frac{V_s(\overline{u}) - \overline{u}}{\rho - 2\sqrt{\overline{u}}} \left(\frac{\frac{t}{R(t)}(2(-\sqrt{\overline{u} - \rho(t - R(t))} + \sqrt{\overline{u}}) - \rho) + \rho}{(-\sqrt{\overline{u} - \rho(t - R(t))} + \sqrt{\overline{u}})^2} + \lambda \right) &\geq 1 \\
\Longleftarrow \; V_s'(\overline{u}) \left(\frac{t(2\sqrt{\overline{u}} - \rho)}{\overline{u}} + 1 \right) &\geq 1 \\
\Longleftarrow \; V_s'(\overline{u}) \left(\min\left\{ \frac{(2\sqrt{\overline{u}} - \rho)}{\sqrt{2\overline{u}}}, \frac{1}{2\rho}(2\sqrt{\overline{u}} - \rho) \right\} + 1 \right) &\geq 1.
\end{aligned}$$

The first implication is due to the facts that $R(t) < 1/\lambda$, and $\frac{2(-\sqrt{\overline{u} - \rho(t' - R(t'))} + \sqrt{\overline{u}}) - \rho)}{(-\sqrt{\overline{u} - \rho(t' - R(t'))} + \sqrt{\overline{u}})^2}$ is decreasing in t'. This can be seen by taking the derivative as in Claim 8 and the fact that $d_1 + d_0 > \Delta\rho$. Thus taking t' such that $\rho(t' - R(t')) = \overline{u}$, and replacing this expression in (21) makes this condition tighter. To see the second implication, note that $t + d_0 = a(t, u) > 0$ which implies $t > \sqrt{\overline{u} - \rho(t - R(t))}$. Note that either $\sqrt{\overline{u} - \rho(t - R(t))} > \sqrt{\overline{u}/2}$ or $\rho(t - R(t)) > \overline{u}/2$ and $t > \overline{u}/(2\rho)$. both expressions in the minimum are uniformly bounded away from 0 because $\sqrt{\overline{u}} > d_1 + d_0 > \rho\Delta$.

[41] The positive root is indicated by the assumption that $a(t, u) \geq t$ in this region.

[42] The negative root is indicated by the assumption that $a(t, u) < t$ in this region.

Claim 10. *$\forall \Delta > 0$, $\exists \lambda_3$ such that $\forall t > 0,\ \forall u > \rho^2/16 + \varepsilon$ with $d_1 + d_0 \leq \rho\Delta$, $\forall \lambda \geq \lambda_3$ (21) holds.*

Proof of Claim: Rewrite (21) as,

$$V_s'(d_1^2)\left(\frac{\frac{t}{\lambda R(t)}(2(d_0+d_1)-\rho)+\rho/\lambda}{(d_0+d_1)^2}+1\right) \geq 1.$$

Note that $\forall \lambda > \tilde{\lambda}$ and $\forall t > 0,\ \forall u > \rho^2/16 + \varepsilon$ with $d_1 + d_0 < \rho\Delta$,

$$\begin{aligned}&\frac{\frac{t}{\lambda R(t)}(2(d_0+d_1)-\rho)+\rho/\lambda}{(d_0+d_1)^2}\\ \geq&\frac{(t/3+2)(2(d_0+d_1)-\rho)}{\Delta^2\rho^2}\\ \geq&\frac{t/3+2}{\Delta^2\rho^2}\left(2Q(t,\rho^2/16+\varepsilon,\tilde{\lambda})-\rho\right),\end{aligned} \tag{24}$$

where,

$$Q(t,u,\tilde{\lambda}) \equiv \left(\min\left\{\sqrt{u}-t+\sqrt{u+t^2+t(\rho-2\sqrt{u})},\sqrt{u+\rho/\tilde{\lambda}}-t+\sqrt{u+t^2+t\left(\rho-2\sqrt{u+\rho/\tilde{\lambda}}\right)}\right\}\right).$$

The first inequality is from the fact that $\frac{t}{\lambda R(t)} > t/3 + 2\ \forall \lambda, t$. To see the last inequality, note first that $d_1 + d_0$ is quasiconcave in $R(t)$: a marginal change in $R(t)$ changes $d_1 + d_0$ by $\frac{1}{2a(t,u)}(1 - \frac{t}{d_1})$, and d_1 is decreasing in $R(t)$. Thus, the two expressions in the minimum replace $R(t)$ with its minimum value of 0 and maximum value of $1/\tilde{\lambda}$. In addition, $Q(t, u, \tilde{\lambda})$ increasing in u $\forall t > 0,\ \forall u > \rho^2/16 + \varepsilon$. To see this, take the derivative:

$$Q_u(t,u,\tilde{\lambda}) \in \left\{\frac{\sqrt{u+t^2+t\,(\rho-2\sqrt{u})}+\sqrt{u}-t}{2\sqrt{u}\sqrt{u+t^2+t\,(\rho-2\sqrt{u})}},\frac{\sqrt{u+t^2+t\left(\rho-2\sqrt{u+\rho/\tilde{\lambda}}\right)}+\sqrt{u+\rho/\tilde{\lambda}}-t}{2\sqrt{u+\rho/\tilde{\lambda}}\sqrt{u+t^2+t\left(\rho-2\sqrt{u+\rho/\tilde{\lambda}}\right)}}\right\}.$$

Both expressions are positive in light of Claim 6. I next show that (24) is uniformly bounded away from 0 across $t > 0$. To see this note that,

$$\lim_{t\to\infty} t\left(2\left(\sqrt{u} - t + \sqrt{u + t^2 + t(\rho - 2\sqrt{u})}\right) - \rho\right) = \rho/4\left(4\sqrt{u} - 1\right)$$
$$\lim_{t\to\infty} t\left(2\left(\sqrt{u + \rho/\tilde{\lambda}} - t + \sqrt{u + t^2 + t\left(\rho - 2\sqrt{u + \rho/\tilde{\lambda}}\right)}\right) - \rho\right) = u - 1/4\left(\rho - 2\sqrt{\rho/\tilde{\lambda} + u}\right)^2$$

Thus both limits are strictly positive for $u > \rho^2/16 + \varepsilon$. Since the expression in (24) is continuous in t and everywhere positive, this implies that it is uniformly bounded away from 0 across $t > 0$.

Thus given the form of (21), the fact that $V_s'(d_1^2) \geq V_s'(\rho^2/16 + \varepsilon)$ by Claim 5, and the fact that $V_s'(\rho^2/16 + \varepsilon) \to 1$ as $\lambda \to \infty$ it suffices to take $\lambda_3 > \tilde{\lambda}$ large enough.

In light of the three claims, taking $\lambda > \lambda_i\ \forall i = 0, 1, 2, 3$ ensures that (21) holds $\forall u > \rho^2/16 + \varepsilon$ and $t > 0$.

Part 2: If for some $\tilde{\lambda}$, $\tilde{V}(t, u) - V_s(u) > 0\ \forall u > \rho^2/16$ and $\forall t \in L_{u,\tilde{\lambda}}$, then $\forall \lambda' < \tilde{\lambda}$, $\tilde{V}(t, u) - V_s(u) > 0\ \forall t \in L_{u,\lambda'}$ and $\forall u > \rho^2/16$.

Recall the definition, $L_{u,\lambda} \equiv \left\{t :\ R(t) + \rho/4 < \sqrt{u + \rho R(t)}\right\}$. Note that $\forall u \geq \rho^2/16$, $L_{u,\lambda} = [0, \bar{t}_{u,\lambda})$, where $\bar{t}_{u,\lambda}$ is increasing in λ, u.

Suppose the conclusion of part 2 is false, and let $\overline{\lambda}$ be the highest witness to this contradiction less than $\tilde{\lambda}$ with associated $\bar{t}$ so that $\tilde{V}(\bar{t}, u) - V_s(u) \not> 0$. This exists because $\tilde{V}$ is differentiable in t, λ. Therefore, also by differentiability of $\tilde{V}$ in t and λ, the following conditions are satisfied at $\overline{\lambda}$ and $\bar{t}$,

$$\tilde{V}(\bar{t}, u) - V_s(u) = 0, \text{ and} \tag{25}$$
$$\frac{d\,\tilde{V}(\bar{t}, u)}{dt} = 0. \tag{26}$$

If (25) did not hold, then a slightly higher λ would also violate the condition. If (26) did not hold then because (25) holds, some nearby t to $\bar{t}$ would have $\tilde{V}(t, u) - V_s(u) < 0$, and then by the same logic one can find a higher λ that also violates the condition. This uses the fact that $t_{u,\lambda}$ is increasing in λ.

It will be helpful to explicitly notate λ in $R(t) \equiv R(t, \lambda)$. Let $k > 0$, and $y \equiv \sqrt{\rho R(\bar{t}, k) + u}$ and $\overline{u} \equiv (y - t)^2 + \rho(s - R(\bar{t}, k))$ be the alternative first action and first threshold loss under

$k = \lambda$. Define the continuing allocation

$$x(t') \equiv \begin{cases} y & t' < \bar{t} \\ d_{\overline{u}}(t' - \bar{t}) + \bar{t} & t' > \bar{t} \end{cases}.$$

Note that for $k = \overline{\lambda}$, $\tilde{V}(\bar{t}, u) - V_s(u) = \int_0^\infty (L^A(t'|x) - D_u(t'))\lambda e^{-\lambda t'} dt'$. Thus,

$$\frac{d(\tilde{V}(\bar{t}, u) - V_s(u))}{d\lambda} \tag{27}$$

$$= \frac{d(\int_0^\infty (L^A(t'|x) - D_u(t'))\lambda e^{-\lambda t})}{d\lambda}\Big|_{\lambda=k=\overline{\lambda}} + \frac{d(\int_0^\infty (L^A(t'|x) - D_u(t')))\overline{\lambda} e^{-\overline{\lambda} t})}{dk}\Big|_{k=\lambda=\overline{\lambda}}$$

$$= \frac{d(\int_0^\infty (L^A(t'|x) - D_u(t')))\lambda e^{-\lambda t})}{d\lambda}\Big|_{\lambda=k=\overline{\lambda}} + \int_0^\infty \frac{d\ L^A(t'|x)}{dk}\Big|_{k=\lambda=\overline{\lambda}} \overline{\lambda} e^{-\overline{\lambda} t} \tag{28}$$

The rest of the proof of part 2 shows that (27) is negative. Because of (25), this contradicts the fact that $\overline{\lambda}$ is the highest witness to a violation of $\tilde{V}(\bar{t}, u) - V_s(u) > 0$ below $\tilde{\lambda}$.

Step 1: The first term in (28) is negative.

Claim 11. *$L^A(t|x) - D_u(t)$ is single crossing from below in $t > 0$.*

Proof of Claim:

$$\frac{d\left(L^A(t|x) - D_u(t)\right)}{dt} = 2(d_u(t) - x(t)).$$

Since $d_u(t)$ is increasing, and $x(t)$ is constant for $t < \bar{t}$, $L^A(t|x) - D_u(t)$ is strictly convex in t for $t < \bar{t}$. Also $d_u(0) - x(0) < 0$ and $L^A(0|x) - D_u(0) = 0$, so $L^A(t|x) - D_u(t)$ is single crossing from below in t for $\bar{t} > t > 0$. Since $L^A(t|x) = D_{\overline{u}}(t - \bar{t})\ \forall t \geq \bar{t}$, Lemma 2 point 3 implies $\mathrm{sign}(L^A(t|x) - D_u(t)) = \mathrm{sign}(\overline{u} - D_u(\bar{t}))$. Continuity of $L^A(t|x)$ in t completes the proof of the claim.

This means that the first term in (28) is of the form $\mathbb{E}[g(t)|t \sim exp(\lambda)]$, where the function g is single crossing from below in t and independent of λ. Note also that an increase in λ corresponds to a downward monotone likelihood ratio shift in the exponential distribution, i.e. for $\lambda' < \lambda''$, $\frac{\lambda' e^{-\lambda' t}}{\lambda'' e^{-\lambda'' t}}$ is increasing in t. Theorem 2 from Athey (2002) delivers that the single crossing property is preserved under monotone likelihood ratio shifts. Because of (25) $\overline{\lambda}$, represents a crossing in this expectation and so its derivative in λ must be negative in order to satisfy single crossing. This completes step 1.

Step 2: The second term in (28) is negative.

First, I rewrite condition (26)

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_0^t \overline{\lambda}(a(t,u)-t')^2 e^{-\overline{\lambda}t'}dt' + e^{-\overline{\lambda}t}V_s(\overline{u}(t,u))\right)\Big|_{t=\overline{t}} \\
=&\frac{\rho R_t(\overline{t},k)}{a(\overline{t},u)}((a(\overline{t},u)-R(\overline{t},k))(1-e^{-\overline{\lambda}\overline{t}}) - \overline{t}V_s'(\overline{u})e^{-\overline{\lambda}\overline{s}}) \\
&+((a(\overline{t},u)-\overline{t})^2 - V_s(\overline{u}(\overline{t},u)))\overline{\lambda}e^{-\overline{\lambda}\overline{t}} + V_s'(\overline{u}(\overline{t},u))(\rho - 2(a(\overline{t},u)-\overline{t})) = 0
\end{aligned}
$$

Decomposing this last equality gives,

$$
\begin{aligned}
&((a(\overline{t},u)-\overline{t})^2 - V_s(\overline{u}(\overline{t},u)))\overline{\lambda}e^{-\overline{\lambda}\overline{t}} + V_s'(\overline{u}(\overline{t},u))(\rho-2(a(\overline{t},u)-t))e^{-\overline{\lambda}\overline{t}}) < 0 \qquad (29)\\
\implies &((a(\overline{t},u)-R(s,k))(1-e^{-\overline{\lambda}\overline{t}}) - \overline{t}V_s'(\overline{u}(\overline{t},u))e^{-\overline{\lambda}\overline{s}}) > 0 \\
\iff &0 > \frac{\rho R_k(s,k)}{a(\overline{t},u)}((a(\overline{t},u)-R(s,k))(1-e^{-\overline{\lambda}\overline{t}}) - sV_s'(\overline{u}(\overline{t},u))e^{-\overline{\lambda}\overline{t}}) \\
\iff &\int_0^\infty \frac{d\ L^A(t'|x)}{dk}\Big|_{k=\lambda=\overline{\lambda}} \overline{\lambda}e^{-\overline{\lambda}t} < 0.
\end{aligned}
$$

The penultimate equivalence follows from $R_k(s,k) \le 0$. The last expression is the second term in (28). Thus, establishing (29) is sufficient to complete the proof of step 2, and thereby part 2. I now simplify (29),

$$
\begin{aligned}
&\left((a(\overline{t},u)-\overline{t})^2 - V_s(\overline{u}(\overline{t},u))\right)\overline{\lambda}e^{-\overline{\lambda}\overline{t}} + V_s'(\overline{u}(\overline{t},u))(\rho-2(a(\overline{t},u)-t))e^{-\overline{\lambda}\overline{t}}) < 0 \\
\iff &\left(\overline{u}(\overline{t},u) - \rho(\overline{t}-R(\overline{t},k)) - V_s(\overline{u}(\overline{t},u))\right)\overline{\lambda}e^{-\overline{\lambda}\overline{t}} + \frac{V_s(\overline{u}(\overline{t},u)) - \overline{u}(t,u)}{\rho - 2\sqrt{\overline{u}(\overline{t},u)}}(\rho - 2(a(\overline{t},u)-\overline{t}))\overline{\lambda}e^{-\overline{\lambda}\overline{t}} < 0 \\
\iff &\frac{V_s(\overline{u}(\overline{t},u)) - \overline{u}(\overline{t},u)}{\rho - 2\sqrt{\overline{u}(\overline{t},u)}}\, 2\left(\sqrt{\overline{u}(\overline{t},u)} - (a(\overline{s},u)-\overline{t})\right) < \rho(\overline{t}-R(\overline{t},k)) \\
\iff &\frac{V_s(\overline{u}(\overline{t},u)) - \overline{u}(s,u)}{\rho - 2\sqrt{\overline{u}(\overline{t},u)}}\, 2\left(\sqrt{\overline{u}(\overline{t},u)} - (a(\overline{t},u)-\overline{t})\right) < \rho(\overline{t}-R(\overline{t},k)) \\
\iff &\frac{V_s(d_1^2) - d_1^2}{\rho - 2d_1}\frac{2}{d_0+d_1} < 1
\end{aligned}
$$

But note that the simplified version of (26) in (21) means that it suffices to prove that,

$$
\begin{aligned}
&\frac{2}{d_0+d_1} < \frac{\frac{\overline{t}}{R(\overline{t})}(2(d_0+d_1)-\rho)+\rho}{(d_0+d_1)^2} + \overline{\lambda} \\
\impliedby &2(d_0+d_1) \ge \rho,
\end{aligned}
$$

which was shown to be true in part 1 since $\bar{t} \in L_{u,\bar{\lambda}}$. This completes part 2.

Part 3: $\forall \lambda > 0$, $\forall u \geq \rho^2/16$, and $t \notin L_{u,\lambda}$, $\tilde{V}(t,u) > V_s(u)$.

Let $u \geq \rho^2/16$, $t \notin L_{u,\lambda}$, and $0 \leq r \leq t$. Define the continuing allocation x at u by

$$x(t') \equiv \begin{cases} d_u(t') & t' < t \\ a(t-r, D_u(r)) + r & r \leq t' < t \\ d_{\overline{u}(t-r,D_u(r))}(t'-t) + t & t' > t \end{cases}$$

That is x_r separates on $[0,r)$, pools on $[r,t)$, and separates again on (t,∞). Note that $L^P(x_0) = \tilde{V}(t,u)$. Define $t - r \equiv \tilde{t}$.

Claim 12.

$$\frac{d\ L^P(x_r)}{dr} \geq 0 \implies \tilde{t} \in L_{D_u(r),\lambda}.$$

Proof of Claim: Let $t - r \equiv \tilde{t}$. By the same logic as the argument for Lemma 13,

$$\begin{aligned} & \frac{d\ L^P(x_r)}{dr} < 0 \\ \iff & \frac{2(a(\tilde{t},u(t)) - R(\tilde{t}))}{a(\tilde{t},D_u(r)) + \sqrt{D_u(r)}}(1-e^{-\lambda\tilde{t}}) - \lambda R(\tilde{t}) \\ & + \left(\frac{2(a(\tilde{t},D_u(r)) - t)}{a(\tilde{t},D_u(r)) + \sqrt{D_u(r)}} - \frac{\lambda t}{1-e^{-\lambda\tilde{t}}}\right) V_s'(\overline{u}(\tilde{t},D_u(r)))e^{-\lambda\tilde{t}} < 0 \\ \Longleftarrow & \frac{2(a(\tilde{t},u(t)) - R(\tilde{t}))}{a(\tilde{t},D_u(r)) + \sqrt{D_u(r)}}(1-e^{-\lambda\tilde{t}}) - \lambda R(\tilde{t}) \leq 0 \end{aligned}$$

where the last implication is because the multiplier of $V_s'(\overline{u}(\tilde{t},D_u(r)))e^{-\lambda\tilde{t}}$ in the third line is less than the last line, and $V_s' > 0$. Suppose that $\tilde{t} \notin L_{D_u(r),\lambda}$, i.e. $\sqrt{D_u(r) + \rho R(\tilde{t})} \leq R(\tilde{t}) + \rho/4$.

$$\begin{aligned} & \frac{2(a(\tilde{t},D_u(r)) - R(\tilde{t}))}{a(\tilde{t},D_u(r)) + \sqrt{D_u(r)}}(1-e^{-\lambda\tilde{t}}) - \lambda R(\tilde{t}) \leq 0 \\ \Longleftarrow & \frac{2(\rho/4)}{R(\tilde{t}) + \rho/4 + \sqrt{D_u(r)}}(1-e^{-\lambda\tilde{t}}) \leq \lambda R(\tilde{t}) \\ \iff & \frac{2(1-e^{-\lambda\tilde{t}})}{\lambda} \leq 4/\rho R(\tilde{t})^2 + R(\tilde{t}) + 4/\rho R(\tilde{t})\sqrt{D_u(r)} \\ \iff & \frac{1-e^{-\lambda\tilde{t}}}{\lambda} \leq 2/\rho R(\tilde{t})^2 + R(\tilde{t}), \end{aligned} \tag{30}$$

where the last line follows from the fact that $L_{\rho^2/16,\lambda} \subset L_{u,\lambda}\ \forall u \geq \rho^2/16$, so $u = \rho^2/16$ is most binding. First, I will show that if (30) holds for some $\tilde{t} = s$, then it also holds $\tilde{t} = s' > s$. To see this assume (30) holds at s. The derivative of the LHS of (30) is less than that of the RHS, i.e.

$$
\begin{aligned}
&\frac{d}{ds}\frac{1-e^{-\lambda s}}{\lambda} \leq \frac{d}{ds}(2/\rho R(s)^2 + R(s)) \\
\Longleftrightarrow\ & e^{-\lambda s} \leq R'(s)(4/\rho R(s) + 1) \\
\Longleftrightarrow\ & e^{-\lambda s} \leq \frac{\lambda e^{-\lambda s}}{1-e^{-\lambda s}}(s - R(s))(4/\rho R(s) + 1) \\
\Longleftarrow\ & \frac{1-e^{-\lambda s}}{\lambda} \leq 4/\rho R(s)^2 + R(s) \\
\Longleftarrow\ & \frac{1-e^{-\lambda s}}{\lambda} \leq 2/\rho R(s)^2 + R(s),
\end{aligned}
$$

where the penultimate line uses $s - R(s) > s/2$ in the exponential distribution. This means that is suffices to prove (30) for the upper boundary of $L_{\rho^2/16,\lambda}$, i.e. $\tilde{t} = \bar{t}_{\rho^2/16,\lambda}$. By definition $R(\bar{t}_{\rho^2/16,\lambda}) = \rho/2$, so (30) reduces to

$$
\frac{1 - e^{-\lambda \bar{t}_{\rho^2/16,\lambda}}}{\lambda} \leq \rho
$$

One can show that $\bar{t}_{\rho^2/16,\lambda} \leq 2\rho/(2 - \lambda\rho)$. Thus it suffices to prove that,

$$
\frac{1 - e^{-2\rho\lambda/(2-\rho\lambda)}}{\lambda} \leq \rho,
$$

This holds because the LHS is equal to ρ at $\lambda = 0$ and decreasing in λ. This completes the proof of the claim.

Note that by Lemma 5, because $u \geq \rho^2/16$, $D_u(r) \geq \rho^2/16\ \forall r \in [0, t]$. This means that $L_{D_u(r),\lambda}$ remains of the form $[0, \bar{t}_{D_u(r),\lambda})$. Let $r^* \in [0, t]$ be the minimum type r with $t - r \in L_{D_u(r),\lambda}$. That is, $\forall r < r^*$, $t - r \notin L_{D_u(r),\lambda}$. By the claim above, $\frac{d\, L^P(x_r)}{dr} \leq 0$ and so $L^P(x_{r^*}) \leq \tilde{V}(t, u)$. Now note that since the continuing allocations are the same on $[0, r^*]$, $L^P(x_{r^*}) - V_s(u) = e^{-\lambda r^*}(\tilde{V}(t - r^*, D_u(r^*)) - V_s(D_u(r^*)))$. Because of parts 1 and 2, and because $t - r^* \in L_{u(t^*),\lambda}$, it holds that $\tilde{V}(t - r^*, D_u(r^*)) \geq V_s(D_u(r^*))$. This means that $\tilde{V}(t, u) \geq V_s(u)$, completing the proof of part 3 and thereby the theorem. *Q.E.D.*

D.4. Proof of Lemma 6

Proof. Note that the pooling continuing allocation is not optimal at $u > \rho^2/16$ by Theorem 2. Assume the pooling allocation is optimal at some $u \leq \rho^2/16$. It must be better than an alternative continuing allocation which introduces a small separating portion at the beginning of the allocation. That is, adapting the condition of Lemma 13 using $\bar{t} = \infty$, $\underline{t} = 0$, and $\tilde{x} = \sqrt{u + \rho/\lambda}$ gives,

$$\begin{aligned} &1 - \frac{2(\sqrt{u+\rho/\lambda} - 1/\lambda)}{\sqrt{u+\rho/\lambda} + \sqrt{u}} \leq 0 \\ \iff &\sqrt{u+\rho/\lambda} - \sqrt{u} \geq 2/\lambda \\ \impliedby &\rho/\lambda \geq 4/\lambda^2 \\ \iff &\lambda\rho \geq 4, \end{aligned}$$

Q.E.D.

D.5. Proof of Corollary 1

Proof.

Claim 13. *Take an arbitrary incentive compatible allocation x. If $\exists \tilde{t} : \underline{\tau}(\tilde{t}) > \rho/8$, then there exists an allocation y that improves on x with the property that y is separating after $\underline{\tau}(\tilde{t})$, i.e. $y(t') = d_{L^A(\tilde{t}|y)}(t) + \tilde{t}\ \forall t \geq \underline{\tau}(\tilde{t})$.*

Proof of Claim: Suppose that for some $t < \tilde{t}$, $x(t) - t > \rho/4$, then clearly for the left adjacent threshold, $x(\underline{\tau}(t)) - \underline{\tau}(t) > \rho/4$. This means that $L^A(\underline{\tau}(t)|x) > \rho^2/16$, which would imply that separating is optimal following this threshold by Theorem 2 and proves the claim. Now suppose that $x(t) - t \leq \rho/4\ \forall t \leq \tilde{t}$. But this means that $L_t^A(t|x) = \rho - 2(x(t) - t) \geq \rho/2\ \forall t < \tilde{t}$. Thus, because $\underline{\tau}(\bar{t}) > \rho/8$, then $L^A(\underline{\tau}(\tilde{t})) > \rho^2/16$ and one can replace x with a separating continuing allocation above $\underline{\tau}(\tilde{t})$, and improve on x by Theorem 2. This proves the claim.

Take a sequence of incentive compatible allocations x_n such that $L^P(x_n) \to L$ where L is the infimum loss of the principal. Take $t_n \equiv \min\{\underline{\tau}_{x_n}(t) :\ t \in T,\ L^A(\underline{\tau}_{x_n}(t)|x_n) \geq \rho^2/16\}$, where $t_n = \infty$ if the relevant set is empty. Without loss of optimality by Theorem 2, replace $\tilde{x}_n(t)$ with

$$y_n(t) \equiv \begin{cases} x_n(t) & t < t_n \\ d_{L^A(t_n|x_n)}(t) + t_n & t \geq t_n \end{cases}.$$

Suppose first there exists a subsequence t_n that converges to some $\bar{t}$. Then the optimal allocation is found by optimizing the allocation on $[0, \bar{t}]$ assuming the separating continuation loss above $\bar{t}$, which exists by Lemma 3 and the fact that the separating continuation loss is continuous in the allocation on $[0, \bar{t}]$. Now suppose there exists a subsequence such that $t_{n_k} \to \infty$, then by the claim $\max\{\underline{\tau}_{y_{n_k}}([0, t_{n_k}))\} \leq \rho/8$ by the claim above. Take a convergent subsequence of $\max\{\underline{\tau}_{y_{n_k}}([0,t_{n_k}))\}$ that converges to some $\bar{t}$. Thus, an optimal allocation is found by optimizing the allocation on $[0,\bar{t}]$ assuming the pooling continuation value above $\bar{t}$. These are exhaustive cases and so an optimum exists.

Now suppose $\rho\lambda \leq 4$. By the claim above, if the optimal allocation has a threshold above $\rho/8$ then it is eventually separating. If it does not, then it is eventually pooling which is suboptimal by Lemma 6.

Q.E.D.

D.6. Proof of Proposition 3

Proof. Note that the inequality in Lemma 13 implies that for x^* optimal, it must be that $x^*(t) > r^*_{x^*}(t)$. Expanding this condition gives $\sqrt{L^A(\underline{\tau}(t)|x^*) + \rho R(\overline{\tau}(t) - \underline{\tau}(t))} > R(\overline{\tau}(t) - \underline{\tau}(t))$. Since the reputation is decreasing in λ, this means that $\forall\lambda$ small, any pooling interval in the continuing allocation at any u has bounded length. Moreover by inspecting this condition, this bound is uniform for small initial losses $L^A(\underline{\tau}(t)|x^*)$. Since the optimal continuing allocation is separating at initial loss $u > \rho^2/16$, this will imply that there exists some M, and $\overline{\lambda} > 0$ such that the optimal continuing allocation $\forall\lambda \leq \overline{\lambda}$ solves,

$$\min_{x\in IC([0,M]:\, L^A(0|x)=u)} \int_0^M (x(t') - t')^2 \lambda e^{-\lambda t'} dt' + e^{-\lambda M} V_s(L^A(M|x)).$$

I complete the proof in three steps. First I extend the result in Theorem 2 to $u \geq \rho^2/36$ in the uniform limit. Second I bound the derivative of the minimized continuation loss in the uniform limit. Third, I show that under this conclusion the solution to (4) always chooses the loss at the first threshold to be greater than $\rho^2/36$.

Step 1: $\forall\varepsilon > 0\ \exists\overline{\lambda} > 0$ such that $\forall\lambda \leq \overline{\lambda}$, $V(u) = V_s(u)\ \forall u \geq \rho^2/36 + \varepsilon$.

Note that $\lim_{\lambda\to 0} V(u) = \rho^2/4\ \forall u \geq 0$. This is because as $\lambda \to 0$ the probability distribution puts zero relative weight on any set of lower types, and so $\lim_{\lambda\to 0} V(u) = \lim_{t\to\infty} L^A(t|x^*) = \rho^2/4$ where x^* is an optimal allocation and the limit is by Lemma 2. This convergence is uniform across $u \in [\rho^2/36 + \varepsilon, \rho^2/16]$ because $V_s(u)$ is increasing in u and less than $\rho^2/4$ in this region.

Let $V_0'(u) \equiv \lim_{\lambda\to 0} V'(u)/\lambda$. Note that by Claim 4 and Theorem 2 $\forall u \geq \rho^2/16$,

$$V_0'(u) = \lim_{\lambda\to 0} V_s'(u)/\lambda = \lim_{\lambda\to 0} \frac{V_s(u) - u}{\rho - 2\sqrt{u}} = \frac{\rho + 2\sqrt{u}}{4}$$

Now recall that if (21) holds for all $t \in [0, M]$ and $u \in [\rho^2/36 + \varepsilon, \rho^2/16]$ then separating is optimal. Take the limit of the LHS of (21) as $\lambda \to 0$. Note that $\lim_{\lambda\to 0} R(t) = t/2$, and let $\underline{d}_1 \equiv \lim_{\lambda\to 0} d_1$ and $\underline{d}_0 \equiv \lim_{\lambda\to 0} d_0$. The limit of the LHS of (21) is given by

$$\lim_{\lambda\to 0}\left[\frac{V_s(d_1^2)-d_1^2}{\rho-2d_1}\left(\frac{\frac{t}{R(t)}(2(d_0+d_1)-\rho)+\rho}{(d_0+d_1)^2}+\lambda\right)\right] = \frac{\rho+2\underline{d}_1}{4}\frac{4(\underline{d}_0+\underline{d}_1)-\rho}{(\underline{d}_0+\underline{d}_1)^2}.$$

As in the proof of Theorem 2 d_1, d_0, and $t/R(t)$ are Lipschitz continuous with a constant independent of λ for small enough λ. Also, as previously stated $V_s(u)$ converges uniformly across u. Thus the above limit converges uniformly on the compact set given by $M \geq t \geq 0$ and $\rho^2/36+\varepsilon \leq u \leq \rho^2/16$. The above expression is greater than 1 if

$$\begin{aligned}
&2(\underline{d}_0+\underline{d}_1)\underline{d}_1+\rho(\underline{d}_0+\underline{d}_1)-\rho/2\underline{d}_1-\rho^2/4 > (\underline{d}_0+\underline{d}_1)^2\\
\iff &\underline{d}_1^2+\rho/2\underline{d}_1+\rho\underline{d}_0-\rho^2/4 > \underline{d}_0^2\\
\iff &\underline{d}_1^2+\rho/2\underline{d}_1+\rho\underline{d}_0-\rho^2/4 > \underline{d}_1^2-\rho t/2\\
\iff &2t+2\underline{d}_1+4\underline{d}_0-\rho > 0
\end{aligned}$$

It can be shown that if $u > \rho^2/36+\varepsilon$ then the LHS is uniformly bounded away from 0 $\forall s \in [0,M]$. This in turn means that $\forall\varepsilon > 0$, $\exists\overline{\lambda} > 0$ such that $\forall\lambda \leq \overline{\lambda}$, $V(u) = V_s(u)$ $\forall u \geq \rho^2/36+\varepsilon$.

Step 2: I will show that $\forall\varepsilon > 0$, $\exists\overline{\lambda} > 0$ such that $\forall u \leq \rho^2/16$, $\forall\lambda < \overline{\lambda}$, $V'(u)/\lambda \leq 3\rho/8+\varepsilon$.

Consider the alternative representation of allocations by their interval partition introduced in Lemma 9, i.e. $x \in IC([0,M])$ is identified with its threshold function $\tau : [0,M] \to [0,M]$. Recall the set of all allowable threshold functions to be $\mathcal{R}$ and that $x_{\tau,u}$ is the unique continuing allocation at u with threshold function $\tau \in \mathcal{R}$ by Lemma 9. With this one can write

$$V(u) = \min_{\tau\in\mathcal{R}}\int_0^M (x_{\tau,u}(t')-t')^2\lambda e^{-\lambda t'}dt' + e^{-\lambda M}V_s(L^A(M|x_{\tau,u})) \tag{31}$$

Thus, by Lemma 9 and the envelope theorem, $V'(u) = \frac{d\,L^P(x_{\tau,u})}{du}$ for some optimal $\tau \in \mathcal{R}$ in (31).

Suppose step 2 does not hold. This means there exists $\varepsilon > 0$, and sequences u_n, λ_n, and $\tau_n \in \mathcal{R}$ such that $\forall n$, τ_n is optimal in (31) under λ_n, u_n, $u_n \leq \rho^2/16$, $\lambda_n \to 0$, and $\frac{d\,L^P(x_{\tau_n,u_n})}{du}/\lambda_n \geq 3\rho/8+\varepsilon$.

Let u_n without loss be the maximum such loss to satisfy the above conditions under λ_n.[43] First consider the case in which there is an infinite subsequence such that a first threshold $t_n > 0$ exists under τ_n. Since $u_n \in [0, \rho^2/16]$ and $t_n \in [0, M]$ are both in compact sets there exists a further subsequence such that $u_n \to \tilde{u}$, and $t_n \to \tilde{t}$. Note that $\tilde{u} \leq \rho^2/36$ from step 1. Otherwise $V(u_n)$ is separating for large enough n and satisfies the bound. Using the definition of $a(t_n, u_n)$ and $\overline{u}(t_n, u_n)$, one can compute,

$$\begin{aligned}&\frac{dL^P(x_{\tau_n,u_n})}{du}/\lambda_n \\ =&\frac{(a(t_n,u_n)-R(t_n))(1-e^{-\lambda_n t_n})}{\lambda_n a(t_n,u_n)}+\frac{(a(t_n,u_n)-t_n)}{a(t_n,u_n)}e^{-\lambda t_n}V'(\overline{u}(t_n,u_n))/\lambda_n > 3\rho/8+\varepsilon \end{aligned} \tag{32}$$

Suppose first that $\tilde{t} = 0$. This, combined with $\tilde{u} \leq \rho^2/36$, means that $u_n < \overline{u}(t_n, u_n) < \rho^2/16$. By the fact that u_n is the maximal violation under λ_n, this means that $V'(\overline{u}(t_n, u_n))/\lambda_n < 3\rho/8 + \varepsilon$. Using this in (32) gives

$$(a(t_n,u_n)-R(t_n)) - \frac{\lambda_n t_n}{e^{\lambda_n t_n}-1} > \lambda_n a(t_n,u_n)3\rho/8.$$

This first term on the LHS goes to $\sqrt{\tilde{u}}$ uniformly for all $\lambda_n > 0$ as $t_n \to 0$. However, the second term on the LHS goes to -1 uniformly for small λ as $t_n \to 0$. Since the RHS is positive and $\sqrt{\tilde{u}} < 1$, this is a contradiction.

Because it is a similar argument, consider now the case in which for some small λ_n there is no first threshold under τ_n. That means there is a sequence of thresholds s_k under τ_n, such that $s_k \to 0$. Take $k > K$ large enough so that $L^A(s_k|x_{\tau_n,u_n}) < \rho^2/16$. This is guaranteed by the approximation in Claim 3. Since u_n was the maximal violation under λ_n, it must be that $V'(L^A(s_k|x_{\tau_n,u_n}))/\lambda_n < 3\rho/8 \; \forall k > K$. Let $\tilde{y} \equiv \sqrt{u_n + s_k(\rho - 2\sqrt{u_n}) + s_k^2} + s_k$. Thus,

$$\begin{aligned}&\frac{dL^P(x_{\tau_n,u_n})}{du}/\lambda_n > 3\rho/8 \\ \iff& \frac{1}{\lambda_n}\int_0^{s_k}\frac{d\,L^A(t'|x_{\tau_n,u_n})}{du}\lambda_n e^{-\lambda_n t'} + e^{-\lambda_n s_k}\frac{d\,L^A(s_k|x_{\tau_n,u_n})}{du}V'(L^A(s_k|x_{\tau_n,u_n}))/\lambda_n > 3\rho/8 \\ \implies& \frac{(\tilde{y}-R(s_k))(1-e^{-\lambda_n s_k}}{\lambda_n \tilde{y}} + e^{-\lambda_n s_k}\frac{\tilde{y}-s_k}{\tilde{y}}V'(L^A(s_k|x_{\tau_n,u_n})) > 3\rho/8 \\ \implies& \tilde{y} - R(s_k) - \frac{\lambda_n s_k}{e^{\lambda_n s_k}-1} > \lambda_n \tilde{y} 3\rho/8.\end{aligned}$$

First note that if the implications above hold then the last line is a contradiction. The reason

[43] $\frac{d\,L^P(x_{\tau,u})}{du}$ is continuous in u by Lemma 10, so this is well defined.

is that as $s_k \to 0$ the LHS goes to $\sqrt{u_n} - 1 < 0$ because $u_n < \rho^2/16$, while the RHS is positive $\forall s_k$. To see the first implication recall the result from Lemma 10 that

$$\frac{d\ L^A(t|x_{\tau_n,u_n})}{du} =$$
$$\frac{x_{\tau_n,u}(t) - t}{x_{\tau_n,u_n}(t) - \underline{\tau}_n(t)} \left(\prod_{(\underline{t},\overline{t}) \in \tilde{J}^p : \overline{t} < t} \frac{x_{\tau_n,u_n}(\underline{t}) - \overline{t}}{x_{\tau_n,u_n}(\underline{t}) - \underline{t}} \right) \left(\prod_{(\underline{t},\overline{t}) \in \tilde{J}^s : \underline{t} < t} e^{2/\rho\left((x_{\tau_n,u_n}(\underline{t}) - \underline{t}) - (x_{\tau_n,u_n}(\max\{\overline{t},t\}) - \max\{\overline{t},t\})\right)} \right).$$

Note that each term in the product is positive for $t' < s_k < \overline{\delta}$ because $x(t') > \overline{\tau}_n(t')$ by definition of $\overline{\delta}$. Thus each term in the product is less than 1 and thereby increasing in $x_{\tau_n,u_n}(t)$ $\forall t$. Thus the fact that $\tilde{y} > x_{\tau_n,u_n}(t) \forall t$, replacing every $x_{\tau_n,u_n}(t)$ by $\tilde{y}$ delivers that

$$\frac{d\ L^A(t|x_{\tau_n,u_n})}{du} < \frac{\tilde{y} - t}{\tilde{y}}.$$

To see why $\tilde{y} > x_{\tau_n,u_n}(t)$, note that by incentive compatibility,

$$\begin{aligned} & (x_{\tau_n,u_n}(s_k^-) - s_k)^2 + \rho(s_k - r^*(s_k^-)) < (x_{\tau_n,u_n}(0) - s_k)^2 + \rho(s_k - r^*(0)) \\ \implies & (x_{\tau_n,u_n}(s_k^-) - s_k)^2 < (\sqrt{u_n + \rho r^*(0)} - s_k)^2 + \rho(s_k - r^*(0)) \\ \implies & x_{\tau_n,u_n}(s_k) < \sqrt{\sqrt{u_n} - \overline{\tau}_x(t))^2 + \rho s_k} + s_k = \tilde{y}. \end{aligned}$$

The first implication is due to $s_k > r^*(s_k^-)$ because s_k is a threshold of x_{τ_n,u_n}. the second implication is due to the fact that the RHS is decreasing in $r^*(0)$.

Now consider the case in which there is an infinite subsequence $t_n \to \tilde{t} > 0$. Then as $n \to \infty$, (32) converges to

$$\frac{(\sqrt{\tilde{u} + \rho\tilde{t}/2} - \tilde{t}/2)\tilde{t}}{\sqrt{u + \rho\tilde{t}/2}} + \frac{\sqrt{\tilde{u} + \rho\tilde{t}/2} - \tilde{t}}{\sqrt{\tilde{u} + \rho\tilde{t}/2}} V_0'(\overline{u}(\tilde{t}, \tilde{u})). \tag{33}$$

Thus it is sufficient to prove that the expression in (33) is bounded by $3\rho/8 + \varepsilon$. Suppose not and first consider the case in which $\overline{u}(\tilde{t}, \tilde{u}) \geq \rho^2/36$. This means the optimal allocation above this first threshold is separating by step 1 and so $V_0'(\overline{u}(\tilde{t}, \tilde{u})) = \frac{\rho + 2\sqrt{\overline{u}(\tilde{t},\tilde{u})}}{4}$. Plugging this into (33) gives

$$\begin{aligned} & \frac{(\sqrt{\tilde{u} + \rho\tilde{t}/2} - t/2)t}{\sqrt{\tilde{u} + \rho\tilde{t}/2}} + \frac{\sqrt{\tilde{u} + \rho\tilde{t}/2} - \tilde{t}}{\sqrt{\tilde{u} + \rho\tilde{t}/2}} V_0'(\overline{u}(\tilde{t}, \tilde{u})) > \rho/3 \\ \iff & \tilde{t}^2/2 - \rho\tilde{t}/3 + \underline{d}_0(\tilde{t} - \rho/12 + \underline{d}_1/2) \geq 0. \end{aligned}$$

By taking the limit of (21) from step 1, because $\tilde{t}$ is optimal given $\tilde{u}$, it must also solve $2\tilde{t} + 2\underline{d}_1 + 4\underline{d}_0 - \rho = 0$. These two equations can be shown to be incompatible.

Now consider that $\overline{u}(\tilde{t}, \tilde{u}) \leq \rho^2/36$ so that $V_0'(\overline{u}(\tilde{t}, \tilde{u})) \leq 3\rho/8 + \varepsilon$ by the assumption that u_n was taken as the maximal violator. Consider first the case in which $a(\tilde{t}, \tilde{u}) - \tilde{t} \geq 0$. But then,

$$\frac{(\sqrt{\tilde{u} + \rho\tilde{t}/2} - \tilde{t}/2)\tilde{t}}{\sqrt{\tilde{u} + \rho\tilde{t}/2}} + \frac{\sqrt{\tilde{u} + \rho\tilde{t}/2} - \tilde{t}}{\sqrt{\tilde{u} + \rho\tilde{t}/2}} 3\rho/8 \geq 3\rho/8$$

But again, this can be shown to be impossible. Lastly, suppose $a(\tilde{t}, \tilde{u}) - \tilde{t} < 0$ which means $\sqrt{\tilde{u} + \rho\tilde{t}/2} < \tilde{t} \implies \tilde{t} > \rho/2$. But then $\overline{u}(\tilde{t}, \tilde{u}) > \rho^2/4$ which is a contradiction. This completes step 2.

Step 3: I will show that there exists $\overline{\lambda} > 0$, and small enough $\varepsilon > 0$, such that the optimal first threshold loss in (4) has $u_1 \equiv (a_1 - t_1)^2 + \rho(t - R(t)) \geq \rho^2/36 + \varepsilon$. Step 1 then provides that the optimal allocation is separating after the first threshold. At an optimal t_1, a_1 in (4) the loss is increasing t_1 if

$$\left((a_1 - t_1)^2 - V(u_1)\right) \lambda e^{-\lambda t_1} + (\rho(1 - R'(t_1)) + 2(t_1 - a_1)) V'(u_1) e^{-\lambda t_1} \geq 0.$$

Fix $u_1 = (a_1 - t_1)^2 + \rho(t - R(t)) < \rho^2/36$. For sufficiently low λ, it can be shown that for any such fixed u_1, if the inequality above holds for some value of a_1 it holds $a_1 = R(t_1)$. That is, setting $\tilde{u}_1 \equiv (t - R(t))^2 + \rho(t - R(t))$, the above inequality implies

$$\left((t_1 - R(t_1))^2 - V(\tilde{u}_1)\right) + (\rho(1 - R'(t_1)) + 2(t_1 - R(t_1))) V'(\tilde{u}_1)/\lambda \geq 0.$$

As $\lambda \to 0$, the above inequality converges uniformly across t_1 to,

$$\begin{aligned} &\left(t_1^2/4 - \rho^2/4\right) + (\rho/2 + t_1)) V_0'(\tilde{u}_1) \geq 0 \\ \implies &\left(t_1^2/4 - \rho^2/4\right) + (\rho/2 + t_1)) 3\rho/8 \geq 0 \\ \iff &t_1 \geq \frac{(\sqrt{13} - 3)\rho}{4}. \end{aligned}$$

But this implies $\tilde{u}_1 \geq \rho^2 \frac{(\sqrt{13}-3)\rho}{8} > \rho^2/36$ a contradiction. *Q.E.D.*

E. Proofs from Section 5

E.1. Proof of Proposition 4

Proof. Take a finite delegation set $\tilde{A} = A_1 \cup ... \cup A_n$ and two different incentive compatible allocations $x, y : T \to \tilde{A}$ such that $x(T) = \tilde{A}$. Note that x satisfies the D1 refinement trivially because there are no off-path actions. First I prove a claim about the implications of the D1 refinement in this context. The next claim says that the off path belief for all unused intermediate actions under y must put probability 1 on a specific type defined below.

Claim 14. *$\forall a \in (\tilde{A} \setminus y(T)) \cap [0, y(M)]$, define $t(a) \equiv \min\{t : \ y(t) > a\}$. If y satisfies the D1 refinement, then $\forall t \in T$, $\forall a \in (\tilde{A} \setminus y(T)) \cap [0, y(M)]$,*

$$L^A(t|y) \leq (a-t)^2 + \rho(t - t(a)).$$

Proof of Claim: Take $a \in \tilde{A} \setminus y(T) \cap [0, y(M)]$. First I prove that for some $t' \in T$

$$\{\tilde{R} \in [0, M] : \ L^A(t'|x) < (a - t')^2 + \rho(t' - \tilde{R})\} \neq \emptyset.$$

If $a \in [0, M)$, then taking $t' = a$ and $\tilde{R} = M$ is sufficient. If instead $a \geq M$, then taking $t' : \ y(t') > a$ and $\tilde{R} = M$ is sufficient.

Note that $\forall \tilde{R} \in [0, M]$ reputation, $L^A(t|y) - \Big((a-t)^2 + \rho(t - \tilde{R})\Big)$ is strictly increasing (decreasing) for $t < (>) t(a)$. This is because by Lemma 8,

$$\begin{aligned}&\frac{d}{dt}\Big(L^A(t|y) - \Big((a-t)^2 + \rho(t-\tilde{R})\Big)\Big)\\ =&2(a - y(t)).\end{aligned}$$

This combined with the fact that $\{\tilde{R} \in [0, M] : \ L^A(t'|x) < (a-t')^2 + \rho(t' - \tilde{R})\}$ is non-empty for some t' and the fact that it is never $[0, M]$ for any t' by incentive compatibility, means that for every $t \neq t(a)$ the condition in (8) holds. This means that the off path belief must put 0 probability on all $t \neq t(a)$, and so in order to satisfy the D1 refinement, it must be that $L^A(t|y) \leq (a-t)^2 + \rho(t - t(a)) \ \forall t \in T$.

Claim 15. *If $A_i \cap y(T)$ is not a singintcharleton then $A_i \cap y(T) = [\min A_i, \tilde{a})$ or $A_i \cap y(T) = [\min A_i, \tilde{a}) \cup \{\overline{a}\}$ for some $\overline{a} > \tilde{a} > \min A_i$.*

Proof of Claim: Suppose A_i is not a singleton. I first prove that for $a < \max A_i \cap y(T)$, it cannot be that $(\underline{s}, \overline{s}) = y^{-1}(a)^{\circ}$ with $\underline{s} < \overline{s}$. Let $y(\overline{s}) \equiv a'' > a$ where $a'' \in y(T) \cap A_i$. Because

A_i is not a singleton, it is an interval, i.e. $(a, a'') \subset A_i$. Note that $r_y(a) = R(\underline{s}, \overline{s}) < \overline{s} \leq r(a'')$ by definition. This means that all $a' \in (a, a'')$ are unused under y, and so by Claim 14 bear reputations $\overline{s}$. Thus for small enough $\varepsilon > 0$, $a + \varepsilon$ gives lower loss than a for types in $(\underline{s}, \overline{s})$.

This means that for all actions lower than the maximum action, the allocation is separating, i.e. $A_i \cap y(T) = [\underline{a}, \tilde{a})$ or $A_i \cap y(T) = [\underline{a}, \tilde{a}) \cup \{\overline{a}\}$ for some $\overline{a} > \tilde{a} > \underline{a}$. To see why $\underline{a} = \min A_i$, note that otherwise $\underline{a} - \varepsilon$ is an unused action under y and by Claim 14 has reputation $y^{-1}(\underline{a})$. Since y is separating on $(y^{-1}(\underline{a}), y^{-1}(\tilde{a}))$, it holds that $\underline{a} > y^{-1}(\underline{a})$, and so for small $\varepsilon > \underline{a} - \varepsilon$ is a profitable deviation for $y^{-1}(\underline{a})$ because it has lower material loss and the same reputation. This proves the claim.

Now take the type $t^* < M$ such that $x(t^*) \neq y(t^*)$, but $x(t') = y(t')$ $\forall t' < t^*$. This is well defined because $\tilde{A}$ is a finite delegation set and because of Claim 15.

Let $\tilde{A}_i \equiv y(\{t' : t' > t^*\}) \cap A_j$ where A_j is the ith indexed interval such that $y(\{t' : t' > t^*\}) \cap A_j \neq \emptyset$. Let $\overline{t}_i^x \equiv \sup\ x^{-1}(\tilde{A}_i)$, $\underline{t}_i^x \equiv \inf x^{-1}(\tilde{A}_i)$, $\overline{t}_i^y \equiv \sup\ y^{-1}(\tilde{A}_i)$, and $\underline{t}_i^y \equiv \inf\ y^{-1}(\tilde{A}_i)$.

Claim 16. $\forall i\ |\tilde{A}_i| = 1$, *and either* $\underline{t}_i^x \leq \underline{t}_i^y$ *and* $\overline{t}_i^x > \overline{t}_i^y$, *or* $\underline{t}_i^x \geq \underline{t}_i^y$ *and* $\overline{t}_i^x < \overline{t}_i^y$.

Proof of Claim: I proceed by induction on i.

Base Step: Let $\max\{y(t^*), x(t^*)\} \equiv a_1$ and $\min\{y(t^*), x(t^*)\} \equiv a_0$. Let $s_x = \sup x^{-1}(a_1)$, $s_y \equiv y^{-1}(a_1)$ and $\tilde{t} = \sup x^{-1}(a_0)$. Note that by the fact that x and y are equal below t^*, $\inf x^{-1}(a_0) = \inf y^{-1}(a_0) \equiv \underline{s}$. Suppose first that $x(t^*) < y(t^*)$. This means that $x^{-1}(a_1) = \underline{t}_1^x \geq \tilde{t} > t^* = \underline{t}_1^y$. I prove that $\overline{t}_1^x < \overline{t}_1^y$. Suppose toward a contradiction that $s_x \geq s_y$. Then by incentive compatibility and condition (M^*),

$$
\begin{aligned}
& L^A(a_0, \tilde{t}|x) \leq L^A(a_1, \tilde{t}|x) \\
\iff & (a_0 - \tilde{t})^2 + \rho(\tilde{t} - R(\underline{s}, \tilde{t})) \leq (a_1 - \tilde{t})^2 + \rho(\tilde{t} - R(\underline{t}_1^x, s_x)) \\
\implies & (a_0 - \tilde{t})^2 + \rho(\tilde{t} - R(\underline{s}, \tilde{t})) \leq (a_1 - \tilde{t})^2 + \rho(\tilde{t} - R(\tilde{t}, s_y)) \\
\implies & (a_0 - \underline{t}_1^y)^2 + \rho(\underline{t}_1^y - R(\underline{s}, \underline{t}_1^y)) < (a_1 - \underline{t}_1^y)^2 + \rho(\underline{t}_1^y - R(\underline{t}_1^y, s_y)) \\
\iff & L^A(a_0, \underline{t}_1^y|y) < L^A(a_0, \underline{t}_1^y|y),
\end{aligned}
$$

where the last line contradicts incentive compatibility of y. Thus $s_x < s_y$. This means that $y^{-1}(t) = (\underline{t}_1^y, s_y)$ where $s_y > \underline{t}_1^y$, and so by Claim 15 $\tilde{A}_1$ is a singleton. This means that $s_x = \overline{t}_1^x$ and $s_y = \overline{t}_1^y$ which proves that $\overline{t}_1^y > \overline{t}_1^x$.

Next suppose that $x(t^*) > y(t^*)$. Since $x(t') = y(t')$ $\forall t' < t^*$, $x(T) = \tilde{A}$, and both allocations are monotonic, $\overline{t}_1^y > t^* = \overline{t}_1^x$. Also $\underline{s} = \underline{t}_1^x = \overline{t}_1^x$.Thus, it only remains to prove that $|\tilde{A}_1| = 1$. But if $\tilde{A}_i$ were a non-degenerate interval then $y(T) \cap [y(t^*), x(t^*)] \neq \emptyset$, but by monotonicity, $x(T) \cap [y(t^*), x(t^*)] = \emptyset$, contradicting the fact that $y(T) \subset x(T)$. This proves the base case.

Inductive Step:

Suppose that $\tilde{A}_i = \{a_i\}$ is a singleton. Let $a_{i+1} \equiv y(\bar{t}_i^y)$.

Suppose first that $\underline{t}_i^x \le \underline{t}_i^y$ and $\bar{t}_i^x > \bar{t}_i^y$. Note that by monotonicity, $\underline{t}_{i+1}^x \ge \bar{t}_i^x > \bar{t}_i^y = \underline{t}_{i+1}^y$. Now assume that $s_y \equiv \sup y^{-1}(y(\underline{t}_{i+1}^y) \le \sup x^{-1}(y(\underline{t}_{i+1}^y) \equiv s_x$. By incentive compatibility of x and condition (M^*),

$$\begin{aligned}
& L^A(a_i, \bar{t}_i^x|x) \le L^A(a_{i+1}, \bar{t}_i^x|x) \\
\iff & (a_i - \bar{t}_i^x)^2 + \rho(\bar{t}_i^x - R(\underline{t}_i^x, \bar{t}_i^x) \le (\bar{t}_i^x - a_{i+1})^2 + \rho(\bar{t}_i^x - R(\underline{t}_{i+1}^x, \bar{s}_x)) \\
\implies & (a_i - \bar{t}_i^x)^2 + \rho(\bar{t}_i^x - R(\underline{t}_i^y, \bar{t}_i^x)) \le (\bar{t}_i^x - a_{i+1})^2 + \rho(\bar{t}_i^x - R(\bar{t}_i^x, \bar{s}_y)) \\
\implies & (a_i - \bar{t}_i^y)^2 + \rho(\bar{t}_i^y - R(\underline{t}_i^y, \bar{t}_i^y)) < (\bar{t}_i^y - a_{i+1})^2 + \rho(\bar{t}_i^y - R(\bar{t}_i^y, \bar{s}_y)) \\
\implies & L^A(a_i \bar{t}_i^y|y) < L^A(a_{i+1}, \bar{t}_i^y|y),
\end{aligned}$$

where the last line violates incentive compatibility. This means that $s_x < s_y$ and so $s_y > \underline{t}_{i+1}^y = \inf y^{-1}(a_{i+1})$. This means that by Claim 15, $\tilde{A}_i$ must be a singleton and so $s_x = \bar{t}_{i+1}^x$ and $s_y = \bar{t}_{i+1}^y$. This completes the inductive step for this case.

Now suppose instead that $\underline{t}_i^x \ge \underline{t}_i^y$ and $\bar{t}_i^x < \bar{t}_i^y$. Note that by definition again, $\underline{t}_{i+1}^y = \bar{t}_i^y$.

Consider first $\tilde{a} \equiv x(\bar{t}_i^x) < a_{i+1}$, i.e. y skips over action $\tilde{a}$. Define $\tilde{s} \equiv \sup x^{-1}(\tilde{a})$. I will first show that $\tilde{s} > \bar{t}_i^y$. Suppose not, i.e. $\tilde{s} \le \bar{t}_i^y$. By Claim 14 and the fact that $\tilde{a}$ is not used under y we have

$$(a_i - \bar{t}_i^y)^2 + \rho(\bar{t}_i^y - R(\underline{t}_i^y, \bar{t}_i^y)) \le (\tilde{a} - \bar{t}_i^y)^2. \tag{34}$$

Note that by incentive compatibility and double use of condition (M^*),

$$\begin{aligned}
& L^A(a_i, \bar{t}_i^x|x) = L^A(\tilde{a}, \bar{t}_i^x|x) \\
\iff & (a_i - \bar{t}_i^x)^2 + \rho(\bar{t}_i^x - R(\underline{t}_i^x, \bar{t}_i^x)) = (\tilde{a} - \bar{t}_i^x)^2 + \rho(\bar{t}_i^x - R(\bar{t}_i^x, \tilde{s})) \\
\implies & (a_i - \tilde{s})^2 + \rho(\tilde{s} - R(\underline{t}_i^x, \tilde{s})) > (\tilde{a} - \tilde{s})^2 \\
\implies & (a_i - \bar{t}_i^y)^2 + \rho(\bar{t}_i^y - R(\underline{t}_i^x, \bar{t}_i^y)) > (\tilde{a} - \bar{t}_i^y)^2 \\
\implies & (a_i - \bar{t}_i^y)^2 + \rho(\bar{t}_i^y - R(\underline{t}_i^y, \bar{t}_i^y)) > (\tilde{a} - \bar{t}_i^y)^2,
\end{aligned}$$

where the last line uses $\underline{t}_i^x \ge \underline{t}_i^y$. The last line contradicts (34) above.

Thus $\tilde{s} > \bar{t}_i^y$. Since by monotonicity $\tilde{s} \le \underline{t}_{i+1}^x$ and $\bar{t}_i^y = \underline{t}_{i+1}^y$, we have $\underline{t}_{i+1}^y < \underline{t}_{i+1}^x$. Now let

$s_x = \sup x^{-1}(a_{i+1})$ and $s_y \equiv \sup y^{-1}(a_{i+1})$. Toward a contradiction, suppose that $s_x \geq s_y$. Again by Claim 14 and the fact that $\tilde{a}$ is not used under y we have

$$(a_{i+1} - \bar{t}_i^y)^2 + \rho(\bar{t}_i^y - R(\bar{t}_i^y, s_y)) \leq (\tilde{a} - \bar{t}_i^y)^2. \tag{35}$$

Again, by incentive compatibility and condition (M^*),

$$\begin{aligned}
& L^A(\tilde{a}, \tilde{s}|x) \leq L^A(a_{i+1}, \tilde{s}|x) \\
\Longleftrightarrow\ & (\tilde{a} - \tilde{s})^2 + \rho(\tilde{s} - R(\bar{t}_i^x, \tilde{s})) \leq (a_{i+1} - \tilde{s})^2 + \rho(\tilde{s} - r_x(a_{i+1})) \\
\Longrightarrow\ & (\tilde{a} - \tilde{s})^2 + \rho(\tilde{s} - R(\bar{t}_i^y, \tilde{s})) \leq (a_{i+1} - \tilde{s})^2 + \rho(\tilde{s} - R(\tilde{s}, s_y)) \\
\Longrightarrow\ & (\tilde{a} - \bar{t}_i^y)^2 < (a_{i+1} - \bar{t}_i^y)^2 + \rho(\bar{t}_i^y - R(\bar{t}_i^y, s_y)).
\end{aligned}$$

The last line contradicts (35). Thus $s_x < s_y$. but this means that $s_y = \sup y^{-1}(a_{i+1}) > s_x \geq \tilde{s} > \inf y^{-1}(a_{i+1}) = \bar{t}_i^y$ which implies that $\tilde{A}_i$ is a singleton and $s_x = \bar{t}_{i+1}^x$ and $s_y = \bar{t}_{i+1}^y$, which completes the inductive step in this case.

Lastly suppose that $x(\bar{t}_i^x) = a_{i+1}$, i.e. $\underline{t}_{i+1}^x = \bar{t}_i^x$. This means that $y(\bar{t}_i^x) = y(\bar{t}_i^y) \equiv a_{i+1}$ Again define $s_x = \sup x^{-1}(a_{i+1})$ and $s_y \equiv \sup y^{-1}(a_{i+1})$. Towards a contradiction suppose that $s_x \leq s_y$. Then by incentive compatibility and condition (M^*)

$$\begin{aligned}
& L^A(a_i, \bar{t}_i^y|y) = L^A(a_{i+1}, \bar{t}_i^y|y) \\
\Longleftrightarrow\ & (a_i - \bar{t}_i^y)^2 + \rho(\bar{t}_i^y - R(\underline{t}_i^y, \bar{t}_i^y)) = (a_{i+1} - \bar{t}_i^y)^2 + \rho(\bar{t}_i^y - R(\bar{t}_i^y, s_y)) \\
\Longrightarrow\ & (a_i - \bar{t}_i^x)^2 + \rho(\bar{t}_i^x - R(\underline{t}_i^y, \bar{t}_i^x)) < (a_{i+1} - \bar{t}_i^x)^2 + \rho(\bar{t}_i^x - R(\bar{t}_i^x, s_y)) \\
\Longrightarrow\ & (a_i - \bar{t}_i^x)^2 + \rho(\bar{t}_i^x - R(\underline{t}_i^x, \bar{t}_i^x)) < (a_{i+1} - \bar{t}_i^x)^2 + \rho(\bar{t}_i^x - R(\bar{t}_i^x, s_x)) \\
\Longleftrightarrow\ & L^A(a_i, \bar{t}_i^x|x) < L^A(a_{i+1}i, \bar{t}_i^x|x).
\end{aligned}$$

The last line contradicts incentive compatibility of x. This means that $s_x > s_y$. Since $s_y \geq \bar{t}_i^y > \bar{t}_i^x$, this means that for j such that $\tilde{A}_i \subset A_j$, A_j is a singleton, which then implies that $\tilde{A}_i$ is a singleton. This means that $\bar{t}_{i+1}^x = s_x$ and $\bar{t}_{i+1}^y = s_y$ completing the inductive step in this case and proving the claim.

Now take the highest action $y(M) \in \tilde{A}_n$. Note that if $\underline{t}_n^x < \underline{t}_n^y$, then by Claim 16 $\bar{t}_n^x > \bar{t}_n^y$ which contradicts that $\bar{t}_n^y = M$. This means that $\underline{t}_n^x \geq \underline{t}_n^y$ and by Claim 16 $\bar{t}_n^x < \bar{t}_n^y$. Let $y(M) \equiv a_n$ and $x(\bar{t}_n^x) \equiv a_{n+1}$, i.e. a_{n+1} is not used under y. Now note that by incentive compatibility and condition (M^*),

$$\begin{aligned}
&L^A(a_n, \bar{t}_n^x|x) = L^A(a_{n+1}, \bar{t}_n^x|x)\\
\iff &(a_n - \bar{t}_n^x)^2 + \rho(\bar{t}_n^x - R(\underline{t}_n^x, \bar{t}_n^x)) = (a_{n+1} - \bar{t}_n^x)^2 + \rho(\bar{t}_n^x - r_x(a_{n+1}))\\
\implies &(a_n - \bar{t}_n^x)^2 + \rho(\bar{t}_n^x - R(\underline{t}_n^y, \bar{t}_n^x)) \geq (a_{n+1} - \bar{t}_n^x)^2 + \rho(\bar{t}_n^x - R(\bar{t}_n^x, M))\\
\implies &(a_n - M)^2 + \rho(M - R(\underline{t}_n^y, M)) > (a_{n+1} - M)^2\\
\iff &L^A(a_{n+1}M|y) > (a_{n+1} - M)^2.
\end{aligned}$$

Notice also that $L^A(t'|y) - (a_{n+1} - t')^2 + \rho(t' - \tilde{R})$ is strictly decreasing in t' by Lemma 8. Thus type M prefers a_{n+1} to their allocation under y for some $\tilde{R}$ reputation and has this preference for a strictly larger set of $\tilde{R}$ reputations than any other type. Thus (8) says that the off path belief puts 0 probability on any type $t \neq M$ following action a_{n+1}. But then the last line in the display is a contradiction to D1 incentive compatibility, completing the argument.

Q.E.D.